

IL NUOVO CIMENTO

RIVISTA INTERNAZIONALE

ORGANO DELLA SOCIETÀ ITALIANA DI FISICA

SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE

E DEL COMITATO NAZIONALE PER L'ENERGIA NUCLEARE

VOL. XLIV B. N. 1

Serie decima

11 Luglio 1966

Singular Hypersurfaces and Thin Shells in General Relativity.

W. ISRAEL

Department of Mathematics, University of Alberta - Edmonton (Alberta)

(ricevuto il 18 Ottobre 1965)

Summary. — An approach to shock waves, boundary surfaces and thin shells in general relativity is developed in which their histories are characterized in a purely geometrical way by the extrinsic curvatures of their imbeddings in space-time. There is some gain in simplicity and ease of application over previous treatments in that no mention of «admissible» or, indeed, any space-time co-ordinates is needed. The formalism is applied to a study of the dynamics of thin shells of dust.

1. — Introduction.

A central problem in gravitational theory, both Newtonian and relativistic, is the formulation of proper junction conditions at surfaces of discontinuity.

These surfaces are of two types: i) *boundary surfaces*, or shock waves, characterized by a jump discontinuity in the density, and ii) *surface layers*, where the density becomes infinite. The Schwarzschild and Oppenheimer-Snyder problems ⁽¹⁾, which require linkage of the interior field of a static or collapsing star to the exterior vacuum field, are well-known examples involving boundary surfaces.

In Newtonian theory, such problems are straightforward. One works in a flat space, in which one can set up a system of co-ordinates which is *a priori* well defined. One has then merely to impose the standard continuity and jump conditions connecting the potential and its first derivatives across the surface.

⁽¹⁾ J. R. OPPENHEIMER and H. SNYDER: *Phys. Rev.*, **56**, 455 (1939); F. HOYLE and J. V. NARLIKAR: *Proc. Roy. Soc.*, A **278**, 465 (1964); C. W. MISNER and D.H. SHARP: *Phys. Rev.*, **136**, B 571 (1964); G. C. MCVITTIE: *Astrophys. Journ.*, **140**, 409 (1964); H. NARIAI and K. TOMITA: *Progr. Theor. Phys.*, **34**, 155 (1965).

In relativity, the ground is more slippery, because the smoothness of the gravitational potentials $g_{\alpha\beta}$ is determined, not only by the smoothness of the physical conditions, but also by the smoothness with which the co-ordinates we happen to be using describe the space-time manifold. We are thus confronted with the not altogether trivial problem of disentangling the bumps arising out of the nature of the physical discontinuity from spurious bumps due to ill-matching of our co-ordinates at the surface in question.

All these matters were already discussed in detail in the pioneering papers of LANCZOS⁽²⁾ more than forty years ago. Since then, singular surfaces have been the subject of a number of studies⁽³⁻⁵⁾. For the special case of boundary surfaces, the problem was placed on an axiomatic basis by LICHNEROWICZ⁽⁴⁾. In this work, the role of the co-ordinates was clearly isolated by the introduction of a special class of «admissible co-ordinates». Nevertheless, these questions are still a recurrent source of confusion in some of the literature.

The aim of part of this paper (Sect. 3, 4, 5, 7) is to present an approach to the known results of LANCZOS, O'BRIEN and SYNGE⁽⁵⁾ and LICHNEROWICZ, in which singular hypersurfaces are characterized in a manifestly invariant, purely geometrical fashion by the extrinsic curvatures of their imbeddings in space-time. By avoiding all reference to space-time co-ordinates in the basic definitions, it is hoped that the exposition gains in simplicity and clarity. The treatment suffers, however, from the limitation that the case where the singular hypersurface is null cannot be included in a natural way, since here the concept of extrinsic curvature breaks down⁽⁶⁾.

A second motivation for this paper relates to the geodesic law of motion. It is known that an idealized «test particle» follows a geodesic of space-time⁽⁷⁾.

⁽²⁾ C. LANCZOS: *Phys. Zeits.*, **23**, 539 (1922); *Ann. der Phys.*, **74**, 518 (1924). See also N. SEN: *Ann. der Phys.*, **73**, 365 (1924).

⁽³⁾ Boundary surfaces: A. H. TAUB: *Ill. Journ. Math.*, **1**, 370 (1957); N. COBURN: *Journ. Math. and Mech.*, **10**, 361 (1961); R. ABRAHAM: *Journ. Math. and Mech.*, **11**, 553 (1962); D. G. B. EDELEN and T. Y. THOMAS: *Journ. Math. Anal. Appl.*, **7**, 247 (1963); G. DAUTCOURT: *Arch. Rat. Mech. and Anal.*, **13**, 55 (1963); H. NARIAI: *Progr. Theor. Phys.*, **34**, 173 (1965). Surface layers: G. DAUTCOURT: *Math. Nachr.*, **27**, 277 (1964). This reference also treats null surface layers (e.g. shells of photons), an important topic which lies outside the scope of the present work.

⁽⁴⁾ A. LICHNEROWICZ: *Théories Relativistes de la Gravitation et de l'Electromagnétisme* (Paris, 1955). Also: J. L. SYNGE: *Relativity: The General Theory* (Amsterdam, 1960), p. 39; W. ISRAEL: *Proc. Roy. Soc.*, A **248**, 404 (1958).

⁽⁵⁾ S. O'BRIEN and J. L. SYNGE: *Commun. Dublin Inst. Adv. Stud.*, A, No. 9 (1952). See also LANCZOS, ref. ⁽²⁾.

⁽⁶⁾ Cf. R. K. SACHS: in *Relativity, Groups, Topology*, edited by DE WITT and DE WITT (New York, 1964), p. 534.

⁽⁷⁾ H. P. ROBERTSON: *Proc. Edinburgh Math. Soc.*, **5**, 63 (1937); L. INFELD and A. SCHILD: *Rev. Mod. Phys.*, **21**, 408 (1949); A. H. TAUB: *Journ. Math. Phys.*, **5**, 112 (1964).

For a particle of finite mass this cannot be true, since the metric becomes singular on the particle's world-line. According to some authors ⁽⁸⁾, such a particle does follow a geodesic of a (variously defined) fictitious « smoothed-out » or « background » metric. However, a certain obscurity surrounds these results, since the prescriptions adopted for separating out the background field do not appear to be co-ordinate-independent. In the present work, a clear-cut answer is obtained to a simpler question. For a cloud of dust, it is easy to see that the metric is regular and the streamlines are geodesics ⁽⁹⁾. If the cloud is compressed into a thin shell, a singularity (in the very mild form of a jump discontinuity) develops. What can now be said about the streamlines? It is shown in Sect. 8 that an infinite number of fictitious regular space-times \tilde{V} can be defined, which are necessarily nonempty (at least in the neighbourhood of the shell) and such that the particles of the shell follow geodesics of \tilde{V} . It is not at present clear whether such results have any but a purely formal interest.

As an application of the general formalism, the paper concludes (Sect. 9) with a derivation of the equation of motion of a spherical shell of dust.

2. – Geometrical preliminaries.

This Section collects the results from differential geometry which will be needed in the sequel.

In a four-dimensional Riemannian manifold V of class C^4 , consider a curve parametrized by t . The absolute derivative of a smooth vector function A defined on this curve will be denoted by $\hat{\partial}A/\hat{\partial}t$. In component form ⁽¹⁰⁾

$$(1) \quad (\hat{\partial}A/\hat{\partial}t)^\alpha \equiv \delta A^\alpha/\delta t \equiv \partial A^\alpha/\partial t + A^\lambda \Gamma_{\lambda\mu}^\alpha dx^\mu/dt.$$

If $A(u, v)$ is defined on the 2-space $x^\lambda = x^\lambda(u, v)$, we have the Ricci commutation relations

$$(2) \quad \left(\left(\frac{\partial}{\partial u} \frac{\partial}{\partial v} - \frac{\partial}{\partial v} \frac{\partial}{\partial u} \right) A \right)^\alpha = R^\alpha_{\beta\lambda\mu} A^\beta \frac{\partial x^\lambda}{\partial u} \frac{\partial x^\mu}{\partial v}.$$

⁽⁸⁾ H. WEYL: *Raum, Zeit, Materie*, 5th edition, Sect. 36 (Berlin, 1923); L. INFELD and J. PLEBANSKI: *Motion and Relativity*, Sect. I-6, (New York, 1960); W. B. BONNOR and N. S. SWAMINARAYAN: *Zeits. Phys.*, **177**, 240 (1964). For critical comments, see M. CARMELI: *Phys. Lett.*, **11**, 24 (1964).

⁽⁹⁾ See e.g. J. L. SYNGE: *Relativity: The General Theory* (Amsterdam, 1960), p. 175.

⁽¹⁰⁾ Throughout the paper Greek indices take the range 1-4, Latin indices the range 2, 3, 4.

Let Σ be a smooth hypersurface in V with unit normal \mathbf{n} :

$$\mathbf{n} \cdot \mathbf{n} = \varepsilon(\mathbf{n}) = \begin{cases} +1 & (\text{spacelike } \mathbf{n}), \\ -1 & (\text{timelike } \mathbf{n}). \end{cases}$$

The expression

$$(3) \quad d\mathbf{s} = \mathbf{e}_{(i)} d\xi^i,$$

for an infinitesimal displacement $d\mathbf{s}$ in Σ in terms of the co-ordinate differentials, determines a natural frame of three linearly independent tangent vectors $\mathbf{e}_{(i)}$ associated with the intrinsic co-ordinates ξ^i . For a tangent vector function \mathbf{A} , intrinsic covariant and contravariant components are defined by

$$(4) \quad \mathbf{A}_i = \mathbf{A} \cdot \mathbf{e}_{(i)}, \quad \mathbf{A} = A^i \mathbf{e}_{(i)}.$$

The intrinsic covariant derivative of \mathbf{A} with respect to ξ^j is the projection of the (nontangential) vector $\partial\mathbf{A}/\partial\xi^j$ onto Σ :

$$(5) \quad \mathbf{A}_{;j} = \mathbf{e}_{(i)} \cdot \partial\mathbf{A}/\partial\xi^j = \partial\mathbf{A}_i/\partial\xi^j - A^h \Gamma_{h,ij},$$

where

$$(6) \quad \Gamma_{h,ij} = \mathbf{e}_{(h)} \cdot \partial\mathbf{e}_{(i)}/\partial\xi^j.$$

The Christoffel symbols $\Gamma_{h,ij}$ can be expressed solely in terms of the metric 3-tensor $g_{ij} = \mathbf{e}_{(i)} \cdot \mathbf{e}_{(j)}$ and its first derivatives. Thus, intrinsic covariant differentiation and the associated intrinsic Riemann 3-tensor R^h_{ijk} do not depend on the nature of the imbedding; for instance, they are invariant under changes of the imbedding space V which preserve the intrinsic metric of Σ .

Properties of a non-intrinsic character enter when we consider the way in which Σ « bends » in V . This is measured by the variations $\partial\mathbf{n}/\partial\xi^i$ of the unit normal. Since each of these three vectors is perpendicular to \mathbf{n} , we can write

$$(7) \quad \partial\mathbf{n}/\partial\xi^i = K_i^j \mathbf{e}_{(j)},$$

thus defining the *extrinsic curvature* 3-tensor K_{ij} of Σ . Explicitly,

$$(8) \quad K_{ij} = \mathbf{e}_{(i)} \cdot \partial\mathbf{n}/\partial\xi^j = -\mathbf{n} \cdot \partial\mathbf{e}_{(j)}/\partial\xi^i = -\mathbf{n} \cdot \partial\mathbf{e}_{(i)}/\partial\xi^j = K_{ji}.$$

The Gauss-Weingarten equations

$$(9) \quad \partial\mathbf{e}_{(i)}/\partial\xi^j = -\varepsilon(\mathbf{n}) K_{ij} \mathbf{n} + \Gamma_{ij}^h \mathbf{e}_{(h)}$$

now follow directly from (6) and (8).

If \mathbf{A} is a vector field tangent to Σ , then according to (4), (5) and (9),

$$(10) \quad \partial \mathbf{A} / \partial \xi^j = A^i{}_{;j} \mathbf{e}_{(i)} - \varepsilon(\mathbf{n}) A^i K_{ij} \mathbf{n}.$$

We now introduce suitable co-ordinates x^α in V , in terms of which the parametric equations of Σ are

$$x^\alpha = f^\alpha(\xi^1, \xi^2, \xi^3),$$

where the f^α are C^4 functions. Taking components of (3) with respect to x^α , one finds

$$(11) \quad e_{(i)}{}^\alpha = \partial x^\alpha / \partial \xi^i$$

From (1) and (11),

$$\partial e_{(i)} / \partial \xi^j = \partial e_{(j)} / \partial \xi^i,$$

a result already used in (8).

Operating on (9) with $\partial / \partial \xi^k$, using (7) and (9) to evaluate the right-hand side, and applying the commutation relations (2) leads to the equations of Gauss and Codazzi:

$$(12) \quad R_{\alpha\beta;\delta} e_{(a)}{}^\alpha e_{(b)}{}^\beta e_{(c)}{}^\gamma e_{(d)}{}^\delta = R_{abcd} + K_{ac} K_{bd} - K_{bc} K_{ad},$$

$$(13) \quad R_{\alpha\beta;\gamma\delta} n^\alpha e_{(b)}{}^\beta e_{(c)}{}^\gamma e_{(d)}{}^\delta = K_{ba;c} - K_{bc;a}.$$

If (12) and (13) are transvected with $g^{bc} g^{ad}$ and g^{bd} respectively, noting that

$$g^{bc} e_{(b)}{}^\beta e_{(c)}{}^\gamma = g^{\beta\gamma} - \varepsilon(\mathbf{n}) n^\beta n^\gamma,$$

the results (expressed in terms of the Einstein tensor $G_{\alpha\beta}$ of V) are

$$(14) \quad -2\varepsilon(\mathbf{n}) G_{\alpha\beta} n^\alpha n^\beta = {}^3R + K_{ab} K^{ab} - K^2,$$

$$(15) \quad G_{\alpha\beta} e_{(a)}{}^\alpha n^\beta = K_a{}^b{}_{;b} - K_{;a}.$$

Here, $K = g^{ab} K_{ab}$, and 3R is the curvature invariant of Σ . Each term of (14) and (15) is a 4-scalar, *i.e.* independent of the co-ordinates x^α .

Finally, we note a result analogous to (10). Let S^{ab} be a 3-tensor field on Σ . Define the associated (discontinuous) 4-tensor

$$(16) \quad S^{\alpha\beta} = S^{ab} e_{(a)}{}^\alpha e_{(b)}{}^\beta \text{ on } \Sigma, \quad S^{\alpha\beta} = 0 \text{ off } \Sigma.$$

Then

$$(17) \quad \nabla_{\beta} S^{\alpha\beta} = e_{(a)}^{\alpha} S^{ab}{}_{;b} - \varepsilon(\mathbf{n}) S^{ab} K_{ab} n^{\alpha},$$

where ∇ denotes the covariant derivative connected with the metric of V .

3. - Singular hypersurfaces.

Let Σ be a 3-space which separates Riemannian space-time into two ⁽¹¹⁾ four-dimensional manifolds V^- , V^+ of class C^4 , each containing Σ as part of its boundary. The unit normal \mathbf{n} (directed from V^- to V^+) will be assumed everywhere spacelike. Let K_{ij}^- , K_{ij}^+ be the extrinsic curvatures of Σ associated with its imbeddings in V^- , V^+ .

If $K_{ij}^- \neq K_{ij}^+$, we shall call Σ a *singular hypersurface of order one*, or, more descriptively, (the history of) a *surface layer*.

If $K_{ij}^- = K_{ij}^+$ everywhere, Σ will be called a *hypersurface of higher order*.

This latter category includes, besides regular hypersurfaces, the class of (histories of) all *boundary surfaces*, characterized by jumps in the energy tensor (*e.g.* the boundary surface separating a star from the surrounding vacuum).

It should be noted that the above definitions make no reference to any four-dimensional space-time co-ordinates.

4. - Boundary surfaces.

The case where Σ is a hypersurface of higher order can be briefly disposed of ⁽³⁻⁵⁾. We obtain from (14) and (15), since their right-hand sides are continuous across Σ ,

$$(18) \quad G_{\alpha\beta} n^{\alpha} n^{\beta}|^- = G_{\alpha\beta} n^{\alpha} n^{\beta}|^+, \quad G_{\alpha\beta} e_{(a)}^{\alpha} n^{\beta}|^- = G_{\alpha\beta} e_{(a)}^{\alpha} n^{\beta}|^+.$$

(We have written $\Psi(P)|^-$ and $\Psi(P)|^+$ for the limits of the function Ψ as the point P on Σ is approached from V^- and V^+ .)

Equations (18) are a generalized form of the *O'Brien-Synge junction conditions* ⁽⁵⁾. Each side of the equations is a 4-scalar and may be evaluated separately (at the same point of Σ) in two independently chosen charts x_{-}^{α} , x_{+}^{α} of V_- , V_+ , which need not match continuously on Σ . That is, given a set of intrinsic co-

⁽¹¹⁾ For simplicity we confine ourselves to the case where there is a single surface of discontinuity. The extension to more general situations is trivial.

ordinates ξ^i , the equations of Σ in each chart,

$$x_-^\alpha = f^\alpha(\xi^1, \xi^2, \xi^3), \quad x_+^\alpha = g^\alpha(\xi^1, \xi^2, \xi^3),$$

must involve C^4 functions f^α and g^α , but equality of these functions is not stipulated.

The relation to Lichnerowicz's approach is discussed in Sect. 8.

5. – Surface layers in vacuo.

If Σ is a surface layer, the 3-tensor

$$\gamma_{ij} = K_{ij}^+ - K_{ij}^-$$

is nonvanishing. The symmetric 3-tensor S_{ij} , defined by the « Lanczos equations »⁽¹²⁾

$$(19) \quad \gamma_{ij} - g_{ij} \gamma_p^p = -\kappa S_{ij}$$

equivalent to

$$(20) \quad \gamma_{ij} = -\kappa(S_{ij} - \frac{1}{2}g_{ij}S), \quad S = S_p^p,$$

will be referred to as the *surface energy tensor* of the layer. (A heuristic justification for this description is given in the Appendix.)

For a surface layer in vacuo, $G_{\alpha\beta} = 0$ in V^- and V^+ . According to (14), (15), this subjects the extrinsic curvatures to the eight conditions

$${}^3R + K_{ab}^\pm K_{\pm}^{ab} - K_{\pm}^2 = 0,$$

$$(K^\pm)_{a;b} - K^\pm_{;a} = 0.$$

Addition and subtraction of these equations yields the equivalent set

$$(21) \quad S^{ab}_{;b} = 0,$$

$$(22) \quad \tilde{K}^a{}^b{}_{;b} - \tilde{K}^a{}_{;a} = 0,$$

$$(23) \quad \tilde{K}_{ab} S^{ab} = 0,$$

$$(24) \quad {}^3R + \tilde{K}_{ab} \tilde{K}^{ab} - \tilde{K}^2 = -\frac{1}{4}\kappa^2(S_{ab} S^{ab} - \frac{1}{2}S^2),$$

(12) $\kappa = 8\pi G/c^2 = 1.86 \cdot 10^{-27}$ cm/g is Einstein's constant of gravitation.

where

$$(25) \quad \tilde{K}_{ab} = \frac{1}{2}(K_{ab}^+ + K_{ab}^-), \quad \tilde{K} = g^{ab} \tilde{K}_{ab}.$$

So far, no four-dimensional co-ordinates have been introduced. Let x_-^α , x_+^α be mutually independent charts of V^- , V^+ which need not join up continuously on Σ . We define the 4-dimensional extensions $S_{\alpha\beta}^\pm$ of S_{ab} into V^- and V^+ by (16). By virtue of (17), the conservation eqs. (21) of the layer can be written

$$(26) \quad e_{(a)}^\alpha \nabla_\beta S_\alpha^\beta |^\pm = 0, \quad n^\alpha \nabla_\beta S_\alpha^\beta |^\pm = -K_{ab}^\pm S^{ab}.$$

Hence, noting (23) and (20),

$$(27) \quad n^\alpha \nabla_\beta S_\alpha^\beta |^+ + n^\alpha \nabla_\beta S_\alpha^\beta |^- = 0,$$

$$(28) \quad n^\alpha \nabla_\beta S_\alpha^\beta |^+ - n^\alpha \nabla_\beta S_\alpha^\beta |^- = \kappa(S_{ab} S^{ba} - \frac{1}{2} S^2).$$

Equations (27), (28) have as their Newtonian analogue the formula

$$\sigma \mathbf{F}^- \cdot \mathbf{n} = -\sigma \mathbf{F}^+ \cdot \mathbf{n} = 2\pi G \sigma^2$$

for the normal components of mechanical force due to self-attraction on the two faces of a plane layer of surface density σ . The tangential force vanishes, according to the first of (26).

6. - Background space.

The preceding results take an interesting form if one imagines Σ imbedded in a fictitious « background space » \tilde{V} such that Σ is regular in \tilde{V} with extrinsic curvature \tilde{K}_{ab} . This prescription does not single out a unique \tilde{V} . Indeed, we are free to contemplate an infinity of different background spaces. On Σ , all of them have the same metric and the same normal components of the Einstein tensor, given by

$$(29) \quad \tilde{G}_{\alpha\beta} n^\alpha n^\beta = \frac{1}{8} \kappa^2 (S_{ab} S^{ab} - \frac{1}{2} S^2), \quad \tilde{G}_{\alpha\beta} e_{(a)}^\alpha n^\beta = 0,$$

according to (14), (15), (22), (24).

From (17), (21) and (23) it follows that in \tilde{V} the conservation laws of the layer resemble the conventional form

$$(30) \quad \tilde{\nabla}_\beta S^{\alpha\beta} = 0.$$

7. – Natural co-ordinates.

Let Σ be an arbitrary timelike hypersurface, with intrinsic co-ordinates ξ^i . Introduce Gaussian co-ordinates⁽¹³⁾ x^α based on Σ by setting $x^i = \xi^i$, $x^1 = \pm$ (geodesic distance normal to Σ) for points in V^\pm respectively. We then have, everywhere in a finite neighbourhood of Σ ,

$${}^4g_{11} \stackrel{*}{=} 1, \quad {}^4g_{i1} \stackrel{*}{=} 0,$$

where $\stackrel{*}{=}$ denotes an equality valid only in special co-ordinates. On Σ ,

$$\begin{aligned} x^1 &\stackrel{*}{=} 0, & {}^4g_{ij} &\stackrel{*}{=} {}^3g_{ij}, \\ n_\alpha &\stackrel{*}{=} \delta^1_\alpha, & e_{(a)}^\alpha &\stackrel{*}{=} \delta_a^\alpha, \\ K_{ij}^\pm &\stackrel{*}{=} -{}^4\Gamma_{ij}^1|^\pm \stackrel{*}{=} \frac{1}{2} \partial {}^4g_{ij} / \partial x^1|^\pm. \end{aligned}$$

From these follow the four-dimensional relations

$$(31) \quad [x^\alpha] \stackrel{*}{=} 0, \quad [g_{\alpha\beta}] \stackrel{*}{=} 0,$$

$$(32) \quad [n_\alpha] \stackrel{*}{=} 0,$$

$$(33) \quad [\partial g_{\alpha\beta} / \partial x^\mu] \stackrel{*}{=} 2[K_{\alpha\beta}] n_\mu \stackrel{*}{=} 2\gamma_{\alpha\beta} n_\mu,$$

$$(34) \quad [I_{\alpha\beta}^{\mu\nu}] \stackrel{*}{=} \gamma_\alpha^\mu n_\beta + \gamma_\beta^\nu n_\alpha - \gamma_{\alpha\beta} n^\mu.$$

Here, $\gamma_{\alpha\beta}$ is the four-dimensional extension of γ_{ij} , defined as in (16), and $[\Phi]$ denotes the jump across Σ , $\Phi^+ - \Phi^-$, for any quantity Φ .

Since $[I_{\alpha\beta}^\mu]$ and $[\partial g_{\alpha\beta} / \partial x^\mu]$ are tensors under C^2 transformations, the validity of (31)–(34) can be immediately extended to a wider class of co-ordinates. Let us define *natural co-ordinates* based on Σ to be any set of co-ordinates which are related to the above Gaussian co-ordinates by a transformation which is C^4 in the interiors of V^- , V^+ (i.e. excluding Σ) and C^2 on Σ . Then (31)–(34) are valid in all natural co-ordinates. The tensor $2\gamma_{\alpha\beta}$ may thus be interpreted as the jump in the normal derivative of $g_{\alpha\beta}$ in natural co-ordinates.

If Σ is a boundary surface, $\gamma_{\alpha\beta} = 0$, and (31), (33) show that, in natural co-ordinates, the metric tensor and its first derivatives are continuous across Σ . This links up our approach with that of LICHNEROWICZ⁽⁴⁾, and shows that natural co-ordinates are identical with Lichnerowicz's « admissible co-ordinates ».

(13) See ref. (9), p. 35.

If Σ is a surface layer, the metric tensor is continuous in natural co-ordinates, and the Lanczos equations [cf. (20), (33)]

$$[\partial g_{\alpha\beta}/\partial x^\mu] \stackrel{*}{=} -2\kappa(S_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}S)n_\mu$$

relate the jumps in the first derivatives to the surface energy tensor of the layer. The Newtonian analogue is Coulomb's formula

$$[\text{grad } V] = 4\pi G\sigma\mathbf{n}$$

for the jump in potential gradient across a layer of surface density σ .

The formulas of this Section, despite their simple appearance, are actually of limited utility, since natural co-ordinates are seldom the most convenient for handling a problem in practice (cf. Sect. 9).

8. - Shells of dust.

If the surface energy tensor has the form

$$(35) \quad S^{ab} = \sigma u^a u^b, \quad u^a u_a = -1,$$

the layer will be referred to as a *shell of dust*. The curves $d\xi^a/ds = u^a$ tangent to Σ are the streamlines of the dust particles.

The conservation equations (21) now reduce to

$$(36) \quad u^a_{;b} u^b = 0,$$

showing that the streamlines are timelike geodesics of Σ , and

$$(\sigma u^a)_{;a} = 0,$$

expressing the conservation of the « nucleonic mass » (*i.e.* the number of particles). From (23),

$$(37) \quad \tilde{K}_{ab} u^a u^b = 0.$$

The 4-accelerations of the dust particles, as measured in V^- and V^+ , are given by

$$\left. \frac{\partial \mathbf{u}}{\partial s} \right|^\pm = \left. \frac{\partial \mathbf{u}}{\partial \xi^j} \right|^\pm \frac{d\xi^j}{ds} = -u^i K_{ij}^\pm u^j \mathbf{n},$$

by virtue of (10) and (36). With respect to independent charts in V^- , V^+ ,

$$[\delta u^\alpha / \delta s + u^a u^b K_{ab} n^\alpha]^\pm = 0 .$$

Hence the equations of motion of the shell can be written

$$(38) \quad e_{(a)}^\alpha \delta u_\alpha / \delta s |^\pm = 0 ,$$

$$(39) \quad n_\alpha \delta u^\alpha / \delta s |^+ + n_\alpha \delta u^\alpha / \delta s |^- = 0 ,$$

$$(40) \quad n_\alpha \delta u^\alpha / \delta s |^+ - n_\alpha \delta u^\alpha / \delta s |^- = \frac{1}{2} \kappa \sigma .$$

Since the individual terms of these equations are 4-scalars, they may be evaluated separately in the mutually independent charts x_+^α , x^α .

From (10), (36) and (37) we can obtain the acceleration of the dust particles as measured in \tilde{V} :

$$\delta u^\alpha / \delta s = 0 .$$

Thus: *the particles of a shell of dust pursue timelike geodesics of any of the fictitious background spaces \tilde{V} introduced in Sect. 6.*

9. - Spherical shell of dust.

The results developed in Sect. 8 will be applied in this Section to derive the equation of motion of a spherical shell of dust.

According to Birkhoff's theorem ⁽¹⁴⁾, the exterior vacuum region V^+ is the Schwarzschild manifold, with metric

$$(ds^2)^+ = f^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - f dt^2 ,$$

where

$$x_+^1 = r, \quad x_+^2 = \theta, \quad x_+^3 = \varphi, \quad x_+^4 = t ,$$

$$f(r) = 1 - 2m/r ,$$

and m is the mass of the shell. The region V^- interior to the shell is flat:

$$(ds^2)^- = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - dT^2 .$$

⁽¹⁴⁾ See ref. (9), p. 276.

The interior and exterior co-ordinates do not join continuously ($T \neq t$ on Σ)⁽¹⁵⁾. However, as already noted, the validity of (38), (39), (40) is unaffected by this circumstance.

Let the equation of the shell be

$$r = R(s),$$

where s is proper time measured along the streamlines. Then

$$\begin{aligned} u_+^\alpha &= \frac{d\omega_+^\alpha}{ds} = (\dot{R}, 0, 0, X) & (\dot{R} = dR/ds), \\ n_\alpha^+ &= (X, 0, 0, -\dot{R}), \end{aligned}$$

where

$$X = [f(R) + \dot{R}^2]^{1/2}/f(R).$$

From (38) (or by differentiating $u_\alpha u^\alpha = -1$), we have

$$0 = u_\alpha \delta u^\alpha / \delta s|_+ = f^{-1} R \left. \frac{\delta^2 R}{\delta s^2} \right|_+ - fX \left. \frac{\delta^2 t}{\delta s^2} \right|_+.$$

This can be used to eliminate $\delta^2 t / \delta s^2|_+$ from

$$n_\alpha \left. \frac{\delta u^\alpha}{\delta s} \right|_+ = X \left. \frac{\delta^2 R}{\delta s^2} \right|_+ - R \left. \frac{\delta^2 t}{\delta s^2} \right|_+,$$

yielding

$$n_\alpha \delta u^\alpha / \delta s|_+ = (fX)^{-1} \delta^2 R / \delta s^2|_+.$$

Now,

$$\delta^2 R / \delta s^2|_+ = d^2 R / ds^2 + \Gamma_{\alpha\beta}^1 u^\alpha u^\beta|_+ = d^2 R / ds^2 + m/R^2,$$

so that finally

$$(41) \quad n_\alpha \delta u^\alpha / \delta s|_+ = (fX)^{-1} (\dot{R} + m/R^2).$$

The corresponding interior value can be written down by formally setting $m = 0$ in (41):

$$(42) \quad n_\alpha \delta u^\alpha / \delta s|_- = (1 + \dot{R}^2)^{-1/2} \dot{R}.$$

⁽¹⁵⁾ That the interior and exterior radial co-ordinates must agree on Σ follows at once from the observations i) that V^- and V^+ induce the same intrinsic metric on Σ and ii) that the co-ordinate lines of θ, φ are tangent to Σ .

Substituting (41) and (42) into (39) we obtain the equation of motion of the shell:

$$\ddot{R}[(1 + \dot{R}^2)^{\frac{1}{2}} + (1 + \dot{R}^2 - 2m/R)^{\frac{1}{2}}] = -m(1 + \dot{R}^2)^{\frac{1}{2}}/R^2.$$

It is easy to verify that a first integral of this equation is

$$(43) \quad (1 + \dot{R}^2)^{\frac{1}{2}} = a + m/(2aR),$$

where a is a constant of integration whose physical meaning will be elucidated.

From (40), (41), (42) we obtain (using relativistic units, in which $\kappa = 8\pi$)

$$4\pi R^2 \sigma = m/a.$$

The left-hand side is the « nucleonic mass », *i.e.* the rest mass of each particle of dust multiplied by the number of particles in the shell. On the other hand, the gravitational mass m gives the mass equivalent of the *total* energy. The difference $m(1 - a)/a$ thus measures the « binding energy » of the shell; *i.e.* the (negative of the) contribution to the shell's gravitational mass due to its kinetic and potential energies.

The further integration of (43) and the detailed description of the motion are dealt with in a following paper ⁽¹⁶⁾.

* * *

It is a pleasure to express my indebtedness to Prof. J. L. SYNGE, who first stimulated my interest in surface layers several years ago.

APPENDIX

By regarding a surface layer as the limit of a layer of uniform finite thickness ε as $\varepsilon \rightarrow 0$, one can adduce a heuristic justification for the name « surface energy tensor » given to the quantities S_{ij} introduced in (19).

Let Σ^- , Σ^+ be the two boundary surfaces separating the finite layer from the vacuum regions V^- , V^+ . In terms of Gaussian co-ordinates based on Σ^- (see Sect. 7) the equations of Σ^- and Σ^+ are $x^1 \stackrel{*}{=} 0$, $x^1 \stackrel{*}{=} \varepsilon$. If K_{ij} denotes the extrinsic curvature of any 3-space $x^1 = \text{const}$, and ${}^3R_{ij}$ its intrinsic Ricci

⁽¹⁶⁾ W. ISRAEL: *Gravitational Collapse of a Spherical Shell and the Problem of Causality*, Preprint Series A, No. 45, Mathematics Department, University of Alberta. (October 1965).

tensor, we have

$$K_{ij} \stackrel{*}{=} \frac{1}{2} \partial^4 g_{ij} / \partial x^1, \quad {}^4 R_{ij} \stackrel{*}{=} \partial K_{ij} / \partial x^1 + Z_{ij},$$

where

$$Z_{ij} = {}^3 R_{ij} - K K_{ij} + 2 K_i^p K_{pj}.$$

Integration of the Einstein field equations

$$R_{\alpha\beta} = -\kappa (T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T),$$

through the layer gives

$$-\kappa \int_0^\varepsilon (T_{ij} - \frac{1}{2} g_{ij} T) dx^1 \stackrel{*}{=} K_{ij}^+ - K_{ij}^- + \int_0^\varepsilon Z_{ij} dx^1.$$

In the passage to the limit $\varepsilon \rightarrow 0$ for fixed K_{ij}^-, K_{ij}^+ , it is reasonable to expect K_{ij} to remain bounded inside the layer. Hence the integral of Z_{ij} tends to zero.

Comparison of the limiting result with (20) shows that

$$S_{ij} \stackrel{*}{=} \lim_{\varepsilon \rightarrow 0} \int_0^\varepsilon T_{ij} dx^1.$$

Thus, S_{ij} is just the integral of Einstein's energy tensor through the thickness of the layer.

RIASSUNTO (*)

Si sviluppa un'approssimazione alle onde d'urto, superfici di contorno e strati sottili in relatività generale in cui le loro storie sono caratterizzate in modo puramente geometrico dalle curvature estrinseche delle loro giaciture nello spazio-tempo. Si ha un qualche guadagno in semplicità e facilità di applicazione rispetto alle trattazioni precedenti in quanto non è necessaria alcuna menzione di « ammissibilità » o, nella fattispecie di alcuna coordinata spazio-temporale. Si applica questo formalismo allo studio della dinamica degli strati sottili di polveri.

(*) Traduzione a cura della Redazione.