

Reflection, Transmission, and Amplification of Sound by a Moving Medium

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Reflection, Transmission, and Amplification of Sound by a Moving Medium*

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The reflection and transmission process is analyzed for plane sound waves originating in air at rest and impinging obliquely on a plane interface with a moving stream. Use of a moving reference frame provides transformation to an equivalent aerodynamic problem of flows past a wavy wall—the rippled interface. The angles of incidence, reflection, and refraction are identified with the Mach angles. The angular relations and the amplitude relations (coefficients of reflection and transmission) are evaluated in closed form. In a graph three zones can be distinguished in the plane of angle of incidence v . Mach number of the moving medium: ordinary reflection and transmission, total reflection, and amplified reflection and transmission. Included are three loci of infinite reflection: i.e., self-excited waves. The energy balance is examined, and the source of amplification is concluded to be the energy of the moving stream. In appendices the results are generalized (1) for the case of two moving media and (2) for differing density and speed of sound in the two media.

INTRODUCTION

SOUND waves entering or leaving obliquely through the sides of a high-speed air jet may experience peculiar reflection and refraction effects.^{1,2} An analysis is made herein for plane waves (not necessarily periodic) impinging on a plane interface. The analysis and the physical interpretation are both simplified by using axes moving with the ripple that must develop in the interface. The acoustic problem is thus changed into the aerodynamic problem of the flows above and below a wavy wall—the rippled interface.

In this view the angles of incidence, reflection, and refraction are regarded as Mach angles in two supersonic streams. The velocity *difference* between these two streams is just the velocity of the original moving stream. The known angle of incidence thereby leads immediately to an equal angle of reflection and a simply-related angle of refraction. The angle of refraction is imaginary when the associated velocity, calculated by the velocity-difference rule, is subsonic. This is a condition of total reflection.

These and other features are developed quantitatively in the analysis that follows. In particular, the discovery of a regime of strong amplification of the reflected and transmitted waves has commanded special attention.

The angular relations between the waves were correctly obtained (with a minor exception, to be noted) in previous work.^{1,2} However, the amplitude relations—coefficients of reflection and transmission—are believed to be in error therein, and no region of amplification was found.

* This research was supported by the Defence Research Board of Canada. An abbreviated nonmathematical account was presented at the Second International Congress on Acoustics, Cambridge, Massachusetts, June 17–23, 1956.

¹ J. B. Keller, *J. Acoust. Soc. Am.* **27**, 1044–1047 (1955).

² P. A. Franken and U. Ingard, *J. Acoust. Soc. Am.* **28**, 126–127 (1956).

SYMBOLS

A	= pressure amplitude
B	= $(M^2 - 1)^{\frac{1}{2}}$
c	= speed of sound
M	= Mach number, U/c
p	= perturbation pressure
R	= reflection coefficient
Re	= real part of
S	= function describing interface
T	= transmission coefficient
t	= time
U	= velocity of upper medium, toward right (original reference frame)
U'	= velocity of lower medium, toward left (ripple-attached reference frame)
U''	= velocity of upper medium, toward left (ripple-attached reference frame)
u	= perturbation velocity component \parallel interface
v	= perturbation velocity component \perp interface
x	= coordinate \parallel interface
y	= coordinate \perp interface
Z	= parameter $(M'^2 B''/M''^2 B')$; = $\sin 2\alpha''/\sin 2\alpha'$ when $ M'' \geq 1$; B'' takes sign of M''
α	= angle between wave normal and mean-interface normal
β	= $ 1 - M^2 ^{\frac{1}{2}}$
ρ	= gas density
ψ	= acoustic intensity
ω	= 2π times frequency
"	= denotes quantities associated with transmitted wave and/or upper medium (U'' or M'' replace U or M , respectively)
'	= denotes quantities associated with reflected wave and/or lower medium (U' or M' replace U or M , respectively).

Absence of primes on p, u, v, α denotes association with incident wave; ρ and c carry primes only when their values differ for the two media.

EQUIVALENT AERODYNAMIC FLOWS

Consider the left-hand sketch in Fig. 1 showing two identical gaseous media: the upper medium is moving, the lower medium is at rest. Plane sound waves are incident obliquely from the lower left. The sound waves can be expected to develop ripples in the interface: if the sound speed is c the ripple speed must be greater than c according to the vector triangle. Thus the ripple moves at supersonic speed relative to the lower medium.

Now consider the upper medium, which is moving with velocity U . The medium partially overtakes the ripple. Hence, the ripple speed relative to this medium is reduced: it corresponds to the shortened solid arrow.

The physical phenomena are independent of whether the ripple moves and the medium is at rest or the medium moves and the ripple is at rest. At the right in Fig. 1 the ripple is shown at rest in a moving frame of reference. The two vectors are wind velocities respectively equal to the ripple velocities reversed. There are thus, in effect, two supersonic flows U' and U'' over a wavy wall. The velocity *difference* in this new reference frame is the same as that in the old

$$U' - U'' = U - 0 = U. \tag{1}$$

On dividing by the speed of sound c ,

$$M' - M'' = M. \tag{2}$$

Both the U' and U'' flows on passing over the corrugations develop Mach waves. But Mach waves are just sound waves in a different terminology—and usually in a moving frame of reference. Thus the upper set constitutes the transmitted sound waves and the lower set the reflected sound waves.

ANGLES OF REFLECTION AND REFRACTION

The angular relations are simple: each of the angles $-\alpha$, incidence; α' , reflection; α'' , refraction—is a Mach angle. The vector triangles yield

$$\left. \begin{aligned} \csc\alpha' &= U'/c = M' \\ \csc\alpha'' &= U''/c = M'' \end{aligned} \right\} \tag{3}$$

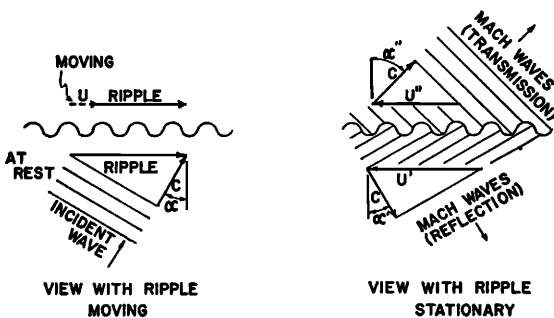


FIG. 1. Sound reflection and transmission by a moving medium in terms of Mach waves generated by ripples running along the boundary.

and the law of reflection

$$\alpha' = \alpha. \tag{4}$$

Equation (2) then yields the law of refraction

$$\csc\alpha'' = \csc\alpha' - M. \tag{5}$$

An exception occurs when the velocity U of the upper medium, measured in the original reference frame, is sufficient to reduce U'' to a subsonic value. Then the corresponding Mach number $|M''| < 1$, and the angle of refraction α'' as defined by Eqs. (3) or (5) is non-existent. A refracted (transmitted) wave of the ordinary kind is absent—according to the further analysis below—and this is a case of total reflection.

COEFFICIENTS OF REFLECTION AND TRANSMISSION

The solution of the aerodynamic problem of the flow over a surface—here, the rippled interface—possessing two-dimensional bumps or waves of small slope is well known. Thus the Mach wave patterns for prescribed contours (not necessarily periodic) can be formulated explicitly.³ The pressure fields in the several waves impinging on or emanating from the rippled interface are of the forms

$$\left. \begin{aligned} \text{incident: } & p = f(x+B'y); & B' &= (M'^2-1)^{\frac{1}{2}} \\ \text{reflected: } & p' = Rf(x-B'y) \\ \text{transmitted: } & p'' = Tf(x+B''y); & B'' &= (M''^2-1)^{\frac{1}{2}} \end{aligned} \right\} \tag{6}$$

Note that B'' takes the sign of M'' .

The associated velocity components normal to the plane of the interface may be written

$$\left. \begin{aligned} \text{incident: } & v = pB'/\rho U' \\ \text{reflected: } & v' = -p'B'/\rho U' \\ \text{transmitted: } & v'' = p''B''/\rho U'' \end{aligned} \right\} \tag{7}$$

Equations (7) relate the (small) local streamline slope (e.g., v''/U'') to the local pressure (e.g., p'') in supersonic flow, according to the Ackeret theory. For the present both Eqs. (6) and (7) will be restricted to supersonic or sonic flow ($|M'|, |M''| \geq 1$).

Boundary Conditions

The unknown reflection and transmission coefficients R and T are to be determined from the boundary conditions at the interface. These are (see Fig. 2)

- (a) the pressure must be equal on both sides

$$p + p' = p''; \tag{8}$$

- (b) the streamline slopes must be equal on both sides (nonzero because the interface is corrugated).

Equating the slopes yields

$$(v + v')/U' = v''/U''. \tag{9}$$

³H. W. Liepmann and A. E. Puckett, *Introduction to Aerodynamics of a Compressible Fluid* (John Wiley and Sons, Inc., New York, 1947), pp. 121-136.

The axial perturbation velocities u , u' , and u'' (not shown in Fig. 2) have been omitted from Eq. (9); the error introduced thereby is of second order. Both conditions (8) and (9) are applied, to a sufficient approximation, at the mean plane $y=0$ of the interface.

The most general statement corresponding to condition (b) is

$$DS/Dt=0 \quad (10)$$

evaluated on each side, where $S(x,y,t)$ is the equation of the interface allowing for deformation with time, and D/Dt is the derivative following the fluid motion. This equation states that the adjacent fluid particles move along the deforming interface. In the ripple-attached reference frame S shows no time dependence (it is effectively stationary and rigid), and Eq. (10) can be shown to reduce to Eq. (9). In the stationary reference frame of previous work^{1,2} Eq. (10) does not reduce to the relation

$$v+v'=v'' \quad (11)$$

employed in that work. For this and other reasons Eq. (11) is believed to be incorrect.

The objection to Eq. (11) can be explained in more physical terms. Compatibility [and Eq. (10)] require that the instantaneous component of resultant velocity—stream plus acoustic—locally normal to the *rippling* interface be the same on both sides—the rippling being caused by the sound waves. Equation (11) in effect implies that the interface is plane, ignoring a component of the stream velocity that comes in due to the sinusoidal surface inclination. The error introduced thereby is not a higher order effect: comparison with Eq. (9) shows that Eq. (11) errs in the value of v'' by a factor U''/U' . The ultimate effects on predicted coefficients of reflection and transmission are very great.

Final Equations

Insertion of Eqs. (6) and (7) with $y=0$ into the boundary conditions Eqs. (8) and (9) yields

$$\begin{aligned} 1+R &= T \\ (1-R)B'/U'' &= TB''/U''^2. \end{aligned}$$

The solution for the coefficients of reflection and transmission is

$$\left. \begin{aligned} R &= \frac{1-Z}{1+Z} = \frac{\text{reflected amplitude}}{\text{incident amplitude}} \\ T &= \frac{2}{1+Z} = \frac{\text{transmitted amplitude}}{\text{incident amplitude}} \end{aligned} \right\} \quad (12)$$

where

$$Z = M''^2 B'' / M''^2 B'. \quad (13)$$

The application of Eqs. (3) together with the definition

of B yields, when α'' exists,

$$Z = \frac{\sin 2\alpha''}{\sin 2\alpha'}. \quad (14)$$

These results were derived for arbitrary (not necessarily periodic) plane sound waves, and for wave incidences providing supersonic or sonic flows in the ripple-attached reference frame: $|M''|$, $|M'| \geq 1$. It can be shown, however, that the results through Eq. (13) still apply for the subsonic case $|M''| < 1$, provided the waves are specialized to be sinusoidal.⁴ In this case Z is imaginary, α'' does not exist, and Eq. (14) does not apply.

TOTAL REFLECTION

For physical interpretation first consider wave incidences such that the upper flow is just sonic: $M'' = \pm 1$ according to Eqs. (3) to (5) (the $-$ sign indicates reversed flow direction). Then $B'' = (M''^2 - 1)^{1/2} = 0$, and Eqs. (13) and (12) yield $Z=0$ and $R=1$. Hence these are cases of total reflection. Between the two values $M'' = \pm 1$ the Mach number M'' of the upper flow is subsonic. It is known that steady subsonic flows radiate no acoustic energy: thus there is no transmitted wave energy. It follows that the entire regime $-1 \leq M'' \leq 1$ corresponds to total reflection.

Actually, a disturbance does penetrate into the upper medium, with amplitude attenuating with distance. If specialization is made to sinusoidal waves, Eq. (6) yields the transmitted pressure in the form

$$p'' = T A e^{i\omega(x+B''y)}.$$

For $|M''| < 1$, $B'' = (M''^2 - 1)^{1/2}$ is replaced by $i(1 - M''^2)^{1/2} = i\beta''$; this gives

$$p'' = T A e^{i\omega x - \omega\beta'' y}. \quad (15)$$

The exponential attenuation for periodic waves is clearly exhibited. For nonperiodic waves different laws of attenuation are obtained.

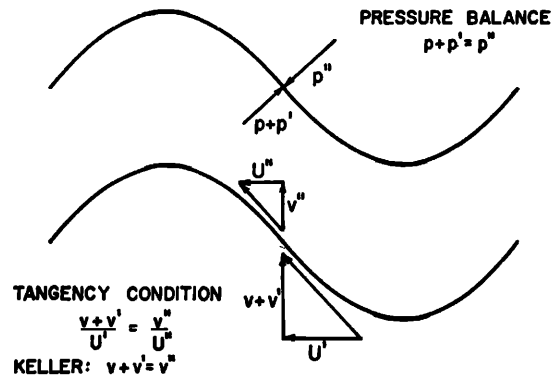


FIG. 2. Conditions of compatibility of the two media at the rippled boundary.

⁴ The basic equations (6) and (7) can be obtained for this case on replacing Eqs. (8.21) of reference 3 by the corresponding complex forms.

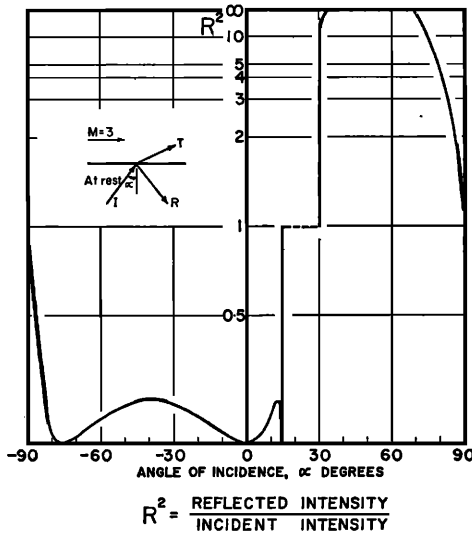


FIG. 3. Reflected energy flux/incident energy flux for sound impinging on a Mach 3 air stream.

A result of the form of Eq. (15) was found by previous investigators, namely, Keller¹ on the one hand, and Franken and Ingard² on the other. They failed to point out, however, that this so-called transmitted wave carries no energy: in the periodic case the pressure and particle velocity are 90° out of phase.

The discussion of energy flow so far has been qualitative; a quantitative discussion is left over to another section.

AMPLIFIED REFLECTION

Figure 3 shows a curve of R^2 (=reflected energy/incident energy) versus angle of incidence for sound impinging on a Mach 3 air stream.⁵ The plateau at $R^2=1$ is the region of total reflection. At larger angles R^2 shoots up to enormous values. This means that the air stream has greatly amplified the reflected sound beam as well as the transmitted beam. There are, in fact, three angles for which the amplification is infinite. One of these cases will now be discussed.

Consider Fig. 4: there exists a certain angle of incidence at which the ripples move at just half the stream speed, if M is above 2. In the figure the lower medium is at rest, the ripple is moving at Mach 1.5, and the upper medium is moving at Mach 3. Thus relative to the upper medium the ripple is moving to the left at Mach 1.5.

Effectively, then, there are two oppositely moving Mach 1.5 flows on opposite sides of the rippled interface. These two flows will generate identical pressure patterns. The pressures will balance on opposite sides of the interface if, and only if, the incident sound wave pattern is removed or made vanishingly weak. This means that the so-called reflected and transmitted (R and T) wave patterns are self-excited disturbances

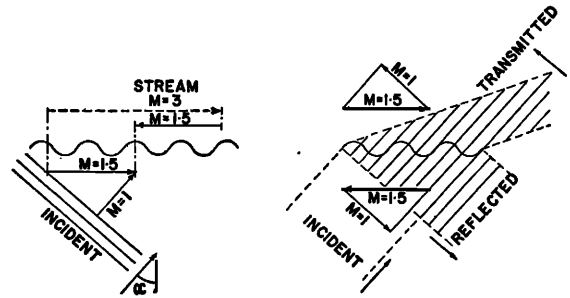


FIG. 4. Construction to illustrate the possibility of self-excited sound waves (infinite coefficients of reflection and transmission). (Bounded beams are shown to demonstrate the difference between boundary refraction and wave-normal refraction; however, modifications of the sound field due to presence of the boundaries, i.e., edge-effects, are not treated in the text.)

feeding, presumably (see below), on energy from the Mach 3 stream. Since in this case the incident wave has vanishing strength, the amplification is effectively infinite.

ENERGY CONSIDERATIONS

Energy Flow through Interface

It will be appropriate now to deal more quantitatively with the question of energy transmission for periodic waves. The acoustic energy flux through unit area of the interface in unit time—that is, the acoustic intensity normal to the interface—can be evaluated on the incident side and on the transmitted side; the two should balance. The intensities are

$$\text{Incident wave plus reflected wave: } \psi_{in} = \langle (p+p')(v+v') \rangle_{av} \quad (16)$$

$$\text{Transmitted wave: } \psi_{out} = \langle p''v'' \rangle_{av} + \rho U \langle u''v'' \rangle_{av} \quad (17)$$

The $\langle \rangle_{av}$ indicates a time average over one cycle. Equation (16) is the usual pressure-particle velocity product correct for a stationary medium. Equation (17) has in addition a second term representing transport of kinetic energy shared by the moving medium and the sound pattern; this equation is derived in Appendix A.

By virtue of Eqs. (6) and (7), Eq. (16) reduces to

$$\psi_{in} = \langle pv \rangle_{av} + \langle p'v' \rangle_{av} \quad (16')$$

If the incident pressure wave is sinusoidal with amplitude A , and R , T and B'' are restricted to real values, then the equations yield

$$\sqrt{2} \langle pv \rangle_{av} = A^2 B' / \rho U'$$

$$\sqrt{2} \langle p'v' \rangle_{av} = -R^2 A^2 B' / \rho U'$$

$$\sqrt{2} \langle p''v'' \rangle_{av} = T^2 A^2 B'' / \rho U''$$

and with $p'' = \rho U'' u''$ (linearized Bernoulli equation),

$$\sqrt{2} \rho U \langle u''v'' \rangle_{av} = T^2 A^2 B'' U / \rho U''^2.$$

The velocities may be divided by c to obtain Mach numbers and $M = U/c$ eliminated by means of Eq. (2).

⁵ Distances along the nonlinear vertical scale are $\sim \tan^{-1} R^2$.

With use of these relations, there result

$$\begin{aligned}\psi_{\text{in}} &= \frac{(1-R^2)A^2B'}{\sqrt{2\rho cM'}} \\ \psi_{\text{out}} &= \frac{T^2A^2B''M'}{\sqrt{2\rho cM''^2}}.\end{aligned}\quad (18)$$

The relation

$$\psi_{\text{in}} = \psi_{\text{out}}$$

is to be tested; it yields

$$1-R^2 = T^2 \frac{B''M'^2}{B'M''^2} = T^2Z. \quad (19)$$

By virtue of Eqs. (12) and (13) this is an identity: the energy flows balance as they should.

With the aid of Eq. (19) the relative energy flux (or intensity) relations may be compared with the relative amplitude relations:

	Amplitude	Intensity	
<i>incident wave:</i>	1	1	}
<i>reflected wave:</i>	R	R^2	
<i>transmitted wave:</i>	T	T^2Z	

(20)

The result that the transmitted wave intensity is proportional to T^2Z , not T^2 , is noteworthy.

For the case of total reflection R and T are complex. The analysis then leads to a generalized version of the energy equation (19) in the form

$$1 - |R|^2 = Re|T|^2Z. \quad (19')$$

Since Z is now imaginary, the right-hand side specifies zero transmitted wave energy. Moreover, the absolute reflection coefficient $|R|$ is unity, so that Eq. (19') is identically satisfied with the reflected energy just balancing the incident energy.

The cases of amplified reflection, $|R| > 1$, $|T| > 2$, correspond to $Z < 0$ in Eqs. (14). (There are three cases of infinite reflection corresponding to the three roots of $Z = -1$.) The energy equation (19) still holds for R greater than unity, the transmitted energy (right-hand side) now being negative. Physically, the transmitted wave, although it is the envelope of *outgoing* cylindrical waves, is carrying energy toward the interface, supplying energy to the reflected wave.

Relationship to Ripple Drag

The source of this energy can be seen to be the moving medium. For example, in the infinite-amplification case of Fig. 4 the upper medium and ripple are both moving to the right, the respective Mach numbers being 3 and 1.5. The ripple acts like a wavy wall and opposes the flow with a certain drag (upper surface only). The medium thus does work on unit area of the wall at a rate (ripple velocity \times drag/unit area); this work rate of the medium may be identified with $-\psi_{\text{out}}$.

The last statement still holds algebraically in the general case, with or without amplification (see Fig. 1). The ripple velocity relative to stationary axes is $U' = U'' + U$ [see Eq. (1)]. Moreover, the average drag/unit area is $\langle \text{pressure} \times \text{slope} \rangle_{\text{av}} = \langle p''v'' \rangle_{\text{av}} / (-U'')$. Thus,

$$-\psi_{\text{out}} = (U'' + U) \langle p''v'' \rangle_{\text{av}} / (-U'') \quad (21)$$

and with $p'' = \rho U''u''$ as before

$$\psi_{\text{out}} = \langle p''v'' \rangle_{\text{av}} + \rho U \langle u''v'' \rangle_{\text{av}}. \quad (17)$$

The last form is identical with Eq. (17), for the energy flux in the transmitted wave. The drag/unit area $\langle p''v'' \rangle_{\text{av}} / (-U'')$ is positive for amplified waves, negative for attenuated waves, with corresponding change in the sign of ψ_{out} .

The reasons for the drag behavior can be found from Fig. 1, showing attenuating conditions, and Fig. 4, showing (infinite) amplification conditions. In Fig. 1 on the right-hand side the flow U'' relative to the ripple is opposite to the stream flow U ; hence here the ripple drag is negative. In Fig. 4 the flow U'' has the same direction as U ; hence here the drag is positive.

Remarks on Mechanism

The mechanism of the energy changes in the upper medium can be explained in terms of nonlinear effects excluded from the first-order theory; these take the form of a cumulative unbalance between the effects of compression and expansion in the wave pattern, with motion of the frame of reference playing a role energy-wise. The energy flow through unit area of the interface is second order, and it can be established that the integral of a third-order increment in energy flow density along an infinite strip normal to the upper stream will yield a second-order result. (Relevant notions may be found in references 6 and 7.) Stated more precisely, the integral refers to the increment in energy carried by the upper medium as measured successively in two unit-width infinite vertical strips separated by unit distance in the stream direction.

An attempt at a quantitative development of this concept has had only partial success. The degree of refinement sufficed to yield an approximate energy balance, supporting the general correctness of the picture: the energy flow increment calculated between the two vertical strips in the upper medium was quite comparable with the energy flow through the interface between the strips. The agreement, however, was not quantitative (the former exceeded the latter by some 40% in the amplified-wave example worked out), and

⁶ W. R. Sears, editor, *General Theory of High Speed Aerodynamics*, Vol. VI of *High Speed Aerodynamics and Jet Propulsion* (Princeton University Press, Princeton, 1954), pp. 386-387; pp. 412-416.

⁷ H. P. Liepmann, "On the relation between wave drag and entropy increase," Rept. No. SM-13726, Douglas Aircraft Company (1950).

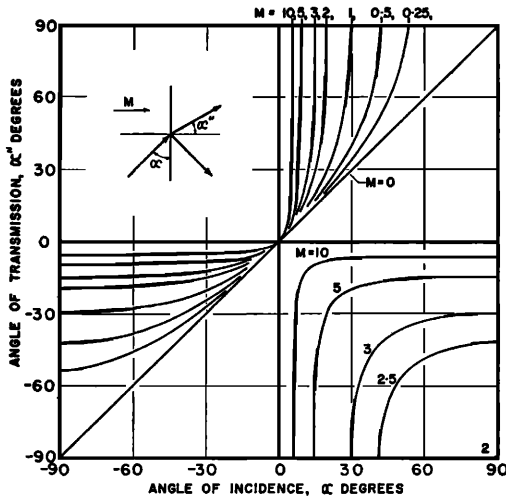


FIG. 5. Angle of refraction α'' vs angle of incidence α for a series of Mach numbers of the moving (second) medium.

in view of this and the complexity the details will not be given.

CALCULATED RESULTS

Curves of angle of refraction α'' vs angle of incidence α for a series of Mach numbers of the moving (second) medium are plotted in Fig. 5; the calculations are based on Eq. (5). The two branches in the curves (for $M > 2$), proceeding respectively from $\alpha = 90^\circ$ and -90° , are noteworthy: the gap between them—along the α axis—is the region of total reflection. (References 1 and 2 incorrectly replace the negative branch by its positive mirror image.)

Contours of constant reflection coefficient R are plotted in Fig. 6 in the plane of angle of incidence vs Mach number. Below the shaded zone there is ordinary reflection ($|R| < 1$). The shaded zone itself represents total reflection ($|R| = 1$). Above the shaded zone there are reflected (and transmitted) waves that are amplified, ($|R| > 1$), the amplification being very high over most of the region.

Two additional features are of interest. It will be noted that the angles of incidence and refraction (compare the vectors I and T) have opposite senses in Fig. 4. This is true whenever the reflected wave is amplified. Also, the wave normals (specified by T) and the edges of the transmitted sound beam (shown dotted) have different directions. Indeed, the directions are different for all incidences and Mach numbers other than zero. This distinction between wave direction and beam direction (of considerable practical importance) does not appear to have been made in earlier work.^{1,2}

CONCLUDING REMARKS

The theory herein has been developed for a rather restricted physical situation for purposes of clarity and simplicity. In Appendix B it is shown how the theory

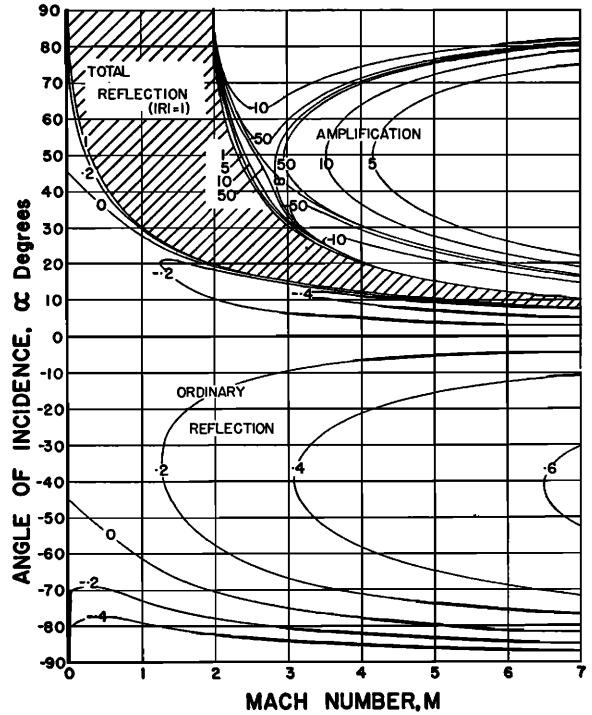


FIG. 6. Contours of constant reflection coefficient R in a graph of angle of incidence vs Mach number of the moving (second) medium.

may be applied when (a) both media are moving and (b) the respective velocity vectors are inclined at an angle to one another. In Appendix C the formulas are generalized to the case where the two media may be different gases, with different densities and speeds of sound.

It is difficult to assess the degree to which the present idealized analysis can apply to physically realizable gas flows. The analysis postulates a mathematically thin surface separating the two relatively moving media. Real flows will develop a mixing region growing in width with distance downstream of the line of initial contact. It seems reasonable to suppose that the analysis will give a good approximation when the sound wavelengths are large compared with the thickness of the mixing region, and may fail in the reverse situation.

APPENDIX A. ACOUSTIC ENERGY FLOW IN A MOVING MEDIUM⁸

The acoustic energy flow in a specified direction in a moving medium is given by the difference in [pressure + (internal energy + kinetic energy)/unit volume] \times (velocity component in specified direction) evaluated with and without the sound field present. The special case of acoustic energy flow normal to the stream direction will be developed in detail to yield Eq. (17). In that case there is no contribution in the absence of the

⁸ Professor K. U. Ingard has shown the author mimeographed class notes of an independent derivation of acoustic energy flow in a moving medium.

sound field, and the energy flow, averaged over a cycle, is

$$\psi_{\text{out}} = \left\langle \left\{ \frac{\gamma}{\gamma-1} (P + \Delta P + p'') + \frac{1}{2} (\rho + \Delta\rho + r'') \right. \right. \\ \left. \left. \times [(U + \Delta U + u'')^2 + (\Delta V + v'')^2] \right\} (\Delta V + v'') \right\rangle_{\text{av}}, \quad (\text{A1})$$

where P, ρ, U are pressure, density, and velocity in the absence of a sound field; $\Delta P, \Delta\rho, \Delta U, \Delta V$ are steady (dc) perturbations due to the sound field, and p'', r'', u'', v'' are the fluctuating (ac) perturbations. ΔV and v'' are directed cross-stream (normal to U and u'').

It can be established that the nonzero dc perturbations are of second order compared with the ac perturbations, taken as first order. Thus, neglect of terms above second order in Eq. (A1) yields

$$\psi_{\text{out}} = \frac{\gamma}{\gamma-1} P \Delta V + \frac{\gamma}{\gamma-1} \langle p'' v'' \rangle_{\text{av}} \\ + \frac{1}{2} U^2 (\rho \Delta V + \langle r'' v'' \rangle_{\text{av}}) + \rho U \langle u'' v'' \rangle_{\text{av}}. \quad (\text{A2})$$

The equation of continuity for the cross-stream direction yields, to second order

$$\langle (\rho + \Delta\rho + r'') (\Delta V + v'') \rangle_{\text{av}} = \rho \Delta V + \langle r'' v'' \rangle_{\text{av}} = 0. \quad (\text{A3})$$

The isentropic equation of state reads

$$r'' = \rho p'' / \gamma P$$

whence

$$\langle r'' v'' \rangle_{\text{av}} = \rho \langle p'' v'' \rangle_{\text{av}} / \gamma P. \quad (\text{A4})$$

The expression for cross-stream acoustic energy flow simplifies to the final form upon substituting Eqs. (A3) and (A4) into (A2):

$$\psi_{\text{out}} = \langle p'' v'' \rangle_{\text{av}} + \rho U \langle u'' v'' \rangle_{\text{av}}. \quad (\text{17})$$

APPENDIX B. TWO MOVING MEDIA

The velocity-difference rule, Eqs. (1) and (2), holds for the more general case when *both* the upper and lower media are moving; in this case U is to be interpreted as the excess of the upper-medium velocity (positive toward the right) over the lower-medium velocity. The case of the *upper* medium at rest is included. This generalization can be inferred from Fig. 1 with the addition of an arbitrary velocity U_1 to both media. A little thought will show that the ripple-velocity vectors, and hence their reverses U' and U'' , are unchanged.

These remarks refer to velocities (or components) in a plane perpendicular to the interface and containing the wave propagation vector (plane of the paper, Fig. 1). For still greater generality, the upper and lower media may possess arbitrary velocity components normal to the plane of the paper (they need not be equal) without affecting any of the equations; this implies that the resultant velocities of the two media may have different directions.

These remarks (and the equations of this paper) do not refer to sound *beam* directions, as distinct from wave-normal directions. The sound-beam directions depend strongly on the motion of the medium containing the sound source, which has herein been assumed stationary.

APPENDIX C. DIFFERENT DENSITIES AND SPEEDS OF SOUND

Let the density and speed of sound in the lower and upper media be ρ', c', ρ'', c'' , respectively. Then a simple generalization of the analysis in the text leads to the following generalization of the main formulas:

$$U' - U'' = U \quad (\text{no change}) \quad (\text{1})$$

$$\left. \begin{aligned} \csc\alpha' &= \frac{U'}{c'} = M' \\ \csc\alpha'' &= \frac{U''}{c''} = M'' \end{aligned} \right\} \quad (\text{3'})$$

$$\alpha' = \alpha \quad (\text{no change}) \quad (\text{4})$$

$$c'' \csc\alpha'' = c' \csc\alpha' - U \quad (\text{5'})$$

$$R = \frac{1-Z}{1+Z}; \quad T = \frac{2}{1+Z} \quad (\text{no change, except the value of } Z): \quad (\text{14})$$

$$Z = \frac{\rho' c'^2 M'^2 B''}{\rho'' c''^2 M''^2 B'} \quad (\text{15'})$$

Eq. (15') simplifies if the two media are the same gas at different density and temperature; for then the ratio $\rho' c'^2 / \rho'' c''^2$ is just $\gamma p' / \gamma p''$ and this must be unity in order that the pressures balance. In this case, then, Eq. (15') for Z reduces to Eq. (15). This result implies that temperature and density differences for the same gas affect the coefficients of reflection R and transmission T only insofar as they affect the angle of refraction α'' [Eqs. (3) and (5)].