

The thermodynamic theory of black holes

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(Communicated by R. Penrose, F.R.S. – Received 26 May 1976)

The thermodynamic theory underlying black hole processes is developed in detail and applied to model systems. It is found that Kerr–Newman black holes undergo a phase transition at $a = 0.68M$ or $Q = 0.86M$, where the heat capacity has an infinite discontinuity. Above the transition values the specific heat is positive, permitting isothermal equilibrium with a surrounding heat bath. Simple processes and stability criteria for various black hole situations are investigated. The limits for entropically favoured black hole formation are found.

The Nernst conditions for the third law of thermodynamics are not satisfied fully for black holes. There is no obvious thermodynamic reason why a black hole may not be cooled down below absolute zero and converted into a naked singularity. Quantum energy-momentum tensor calculations for uncharged black holes are extended to the Reissner–Nordstrom case, and found to be fully consistent with the thermodynamic picture for $Q < M$. For $Q > M$ the model predicts that ‘naked’ collapse also produces radiation, with such intensity that the collapsing matter is entirely evaporated away before a naked singularity can form.

1. INTRODUCTION

Classical thermodynamics, much less statistical mechanics, fails to cope properly with self-gravitating systems. The tendency for a gravitating fluid to implode and grow structure is a fundamental asymmetry of the universe, but its description in terms of a natural progress towards equilibrium and maximum entropy is impaired by the circumstance that catastrophic collapse of matter can occur, for which no equilibrium end state is possible. Gravitational collapse can apparently lead to a state of unbounded entropy and space curvature. Nevertheless it has long been appreciated that the presence of an event horizon around such catastrophic collapse does lead to an equilibrium end state of sorts – the black hole. The entropy of a classical black hole is still unbounded though, making the equilibrium analogy appear a little strained. However, the inclusion of quantum effects (Hawking 1975) damps out this divergence and assigns a definite entropy to the black hole (which diverges as $\hbar \rightarrow 0$). The radical departure is that this entropy has a *geometric* definition through the general theory of relativity. With the entropy is associated a whole range of thermodynamic quantities. Indeed, the black hole system is thermodynamically completely determined, with an explicit fundamental equation, which is more than can be said for most thermodynamic systems. Other equilibrium end

states may also exist. As yet, no progress has been made along these lines on more general, non-equilibrium gravitating systems, for which traditional equilibrium thermodynamic techniques are inappropriate.

The relation between the geometrically defined thermodynamic quantities and their traditional counterparts is still obscure. Nevertheless, it is possible to apply traditional thermodynamic techniques to at least one type of gravitational system, namely the Kerr–Newman family of black holes. Most of the familiar thermodynamic concepts have close analogues in the black hole case, and in particular it is known that the first and second laws may be applied if one makes the identification

entropy $S \propto$ area of event horizon A ,

temperature $T \propto$ surface gravity κ .

The fact that a quantity κ which is constant over the event horizon may be defined at all is equivalent to the zero law of thermodynamics (Carter 1974).

The consequences of these thermodynamic properties are investigated in §§2 and 3. In §4 the third law is shown to be not fully consistent with the black hole situation. The third law is important because it is bound up with the question of the unattainability of the absolute zero of temperature. In the Kerr–Newman case, this turns out to be equivalent to the non-existence of naked singularities. This prompts an investigation into what would happen during the collapse of a massive body with an electric charge greater than its mass. It is very fortunate that this problem may be readily treated using the two dimensional model for calculation of the energy-momentum tensor of a quantum field situated in the curved space background of the collapsing matter. The results predict that this type of object also radiates (though not with a thermal spectrum) and in fact the radiation is so intense that the collapsing matter is completely radiated away before the naked singularity has had time to form. This suggests a quantum version of the cosmic censorship hypothesis.

The full implication of the gravitational–thermodynamic connection is not yet apparent. Classical gravity coupled to quantum matter inevitably endows gravity with some quantum aspects. For example, the expectation value for the energy flux out of a black hole is also the expectation value for the mass, and hence the geometry, of the black hole itself. The formal scheme

$$G_{\mu\nu} = \langle T_{\mu\nu} \rangle$$

for Einstein's equations bears some similarity to the semi-classical treatment of radiation in the early days of quantum mechanics. It was then possible, by invoking thermodynamic arguments, to deduce quantum features of the electromagnetic field by quantising only the matter which is coupled to it. In the same way, it should be possible to deduce, via thermodynamic arguments, quantum properties of the gravitational field by investigating only the quantum nature of the matter fields which are coupled to it ('quantum gravity without quantum gravity'?).

2. BASIC THEORY

The fundamental thermodynamic relation, which contains all the information about the thermodynamic state of black hole matter, has been given by Smarr (1973)

$$M^2 = \frac{1}{4}(A/4\pi) + (4\pi/A)(J^2 + \frac{1}{4}Q^4) + \frac{1}{2}Q^2, \tag{2.1}$$

where J and Q are the angular momentum and electric charge on the black hole, respectively. This expression is correct even for classical black hole systems, for which it was originally devised. New interest attaches to (2.1) with the identification of A with a real entropy

$$S = \frac{1}{4}kA, \tag{2.2}$$

where k is Boltzmann's constant. We may then regard the parameters S, J and Q as a complete set of global quantities for black hole matter, $M = M(S, J, Q)$. In ordinary thermodynamics, such parameters are called 'extensive' because they may be added together for composite systems. Usually, the total energy is a homogeneous first order function of the global parameters. However (2.1) is not homogeneous first order in S, J and Q . This is because black hole matter cannot be 'divided' up into subsystems with a separate identity. Although the areas of two separate black holes are additive, the sum of the areas of two black holes of masses M_1 and M_2 is always less than the area of the black hole obtained by coalescing these two together ($M_1^2 + M_2^2 \leq (M_1 + M_2)^2$) without loss of total energy. This is an expression of the fact that the coalescence of identical black holes is itself an irreversible process (unlike identical laboratory systems which may be coalesced without thermodynamic change). Nevertheless, we shall continue to call S, J and Q 'extensive' parameters. The fundamental equation is therefore

$$M = M(S, J, Q) = (2S + (1/8S)(J^2 + \frac{1}{4}Q^4) + \frac{1}{2}Q^2)^{\frac{1}{2}}. \tag{2.3}$$

It has been convenient to choose units to make the value of k to be $1/8\pi$. Equation (2.3) is often useful in the inverted form

$$S = \frac{1}{4}M^2 - \frac{1}{8}Q^2 + \frac{1}{4}M^2[1 - Q^2/M^2 - J^2/M^4]^{\frac{1}{2}}. \tag{2.4}$$

The first law of thermodynamics states that the total energy in any thermodynamic change is conserved. If M changes by an infinitesimal quantity dM , then

$$dM = T dS + \Omega dJ + \Phi dQ, \tag{2.5}$$

where the quantities T, Ω and Φ are defined by

$$T = \frac{\partial M}{\partial S} = M^{-1} \left[1 - \frac{J^2 + \frac{1}{4}Q^4}{16S^2} \right], \tag{2.6}$$

$$\Omega = \frac{\partial M}{\partial J} = \frac{J}{8MS}, \tag{2.7}$$

$$\Phi = \frac{\partial M}{\partial Q} = \frac{Q(Q^2 + 8S)}{16MS}. \tag{2.8}$$

These are the corresponding 'intensive' parameters (constant everywhere on the horizon). Ω is the angular velocity of the event horizon associated with the angular momentum J and Φ is the electric potential associated with the electric charge Q . The relations (2.7) and (2.8) are well known classical black hole relations. T is the temperature associated with the entropy S . The relation (2.6) was found by Hawking (1975) from quantum theory. Equations (2.6)–(2.8) constitute three equations of state (when M has been eliminated by using (2.3)).

If we regard M as a function of S , J and Q^2 instead, then (2.3) is homogeneous of degree $\frac{1}{2}$ in the extensive parameters. Euler's theorem then requires

$$\frac{1}{2}M = TS + \Omega J + \Theta Q^2 = TS + \Omega J + \frac{1}{2}\Phi Q \quad (2.9)$$

where

$$\Theta = \frac{\Phi}{2Q} = \frac{Q^2 + 8S}{32MS}.$$

This relation was originally noted by Smarr (1973). It is the black hole version of the famous Gibbs–Duhem relation of thermodynamics.

Differentiating (2.9) and using the first law (2.5) we get

$$-\frac{1}{2}dM = SdT + Jd\Omega + Q^2d\Theta, \quad (2.10)$$

whence

$$\left. \begin{aligned} (\partial M/\partial T)_{\Omega, \Theta} &= -2S, \\ (\partial M/\partial \Omega)_{T, \Theta} &= -2J, \\ (\partial M/\partial \Theta)_{\Omega, T} &= -2Q^2. \end{aligned} \right\} \quad (2.11)$$

3. THE PHASE TRANSITION

Great importance is attached to the various specific heats of thermodynamic systems. These may be defined per unit mass, but it is more appropriate here to consider the full thermal capacities:

$$C_x = T(\partial S/\partial T)_x$$

with some set of parameters, denoted by x , held constant.

Suppose a rotating black hole is held in equilibrium at some temperature T in a temperature bath (see §4). If the external thermal bath is increased in temperature slightly, the black hole will absorb energy reversibly. This absorption will be isotropic, and will not change the angular momentum J , or the charge Q , on average. To compute $C_{J, Q}$ we eliminate M between equations (2.4) and (2.6), and differentiate with J and Q held constant. The result is, after some simplification:

$$C_{J, Q} = T \left(\frac{\partial S}{\partial T} \right)_{J, Q} = \frac{8MS^3T}{J^2 + \frac{1}{4}Q^4 - 8T^2S^3}. \quad (3.1)$$

For a Schwarzschild black hole, $J = Q = 0$ and C reduces to $-M/T \equiv -1/T^2 \equiv -M^2$. This quantity is negative definite, which reflects the well known fact that Schwarzschild black holes get hotter as they radiate energy.

To examine the opposite extreme, use (2.6) to rewrite (3.1) as

$$C_{J,Q} = \frac{MST}{2 - T(2M + ST)}. \tag{3.2}$$

Then as the extreme Kerr–Newman case is approached, $T \rightarrow 0$ (see §5), $S \rightarrow \frac{1}{4}M^2 - \frac{1}{8}Q^2 > 0$ so $C_{J,Q} \rightarrow 0+$. $C_{J,Q}$ has clearly changed sign before this limit.

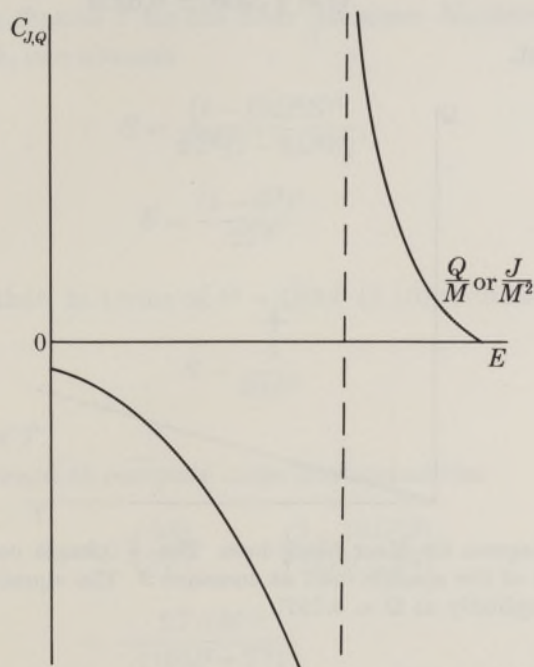


FIGURE 1. General behaviour of heat capacity at constant J and Q . For low values of J/M^2 and Q/M it is negative. The broken line indicates the position of the phase transition, at which $C_{J,Q}$ suffers an infinite discontinuity. Beyond this line it is positive, falling to zero at the extreme Kerr–Newman value E , which is the thermodynamic limit.

It is possible to express $C_{J,Q}$ entirely as a function of J , Q and M by eliminating S and T from (3.1) using (2.6) and (2.9). The general behaviour of $C_{J,Q}$ as a function of J or Q , at fixed mass M , is shown in figure 1. The most conspicuous feature is the infinite discontinuity at some values of J and Q , after which the specific heat changes from negative to positive.

The point at which the discontinuity occurs is for $J^2 = \alpha M^4$, $Q^2 = \beta M^2$ for some values of α and β . To determine the possible values, substitute for J and Q into (2.3) and (2.6) with the restriction (from (3.1)) that $J^2 + \frac{1}{4}Q^4 = 8T^2S^3$. It is then possible to eliminate S and T entirely, to obtain an equation for α and β

$$\alpha^2 + 6\alpha + 4\beta = 3. \tag{3.3}$$

We must also impose the restriction $\alpha, \beta > 0$ and $\alpha + \beta < 1$ (i.e., $T > 0$). The case of an uncharged black hole is $\beta = 0$, whence

$$\alpha = 2\sqrt{3} - 3 \simeq 0.464.$$

The ratio J/M is usually called a , so

$$a \simeq 0.68M \quad (3.4)$$

at the critical point.

For $\beta > 0$, $a < 0.68M$, and in the limit of $J = 0$ (Reissner-Nordstrom black hole) $\alpha = 0$ and

$$\beta = \frac{3}{4}$$

or

$$Q = \frac{1}{2}\sqrt{3M} \simeq 0.86M \quad (3.5)$$

at the critical point.

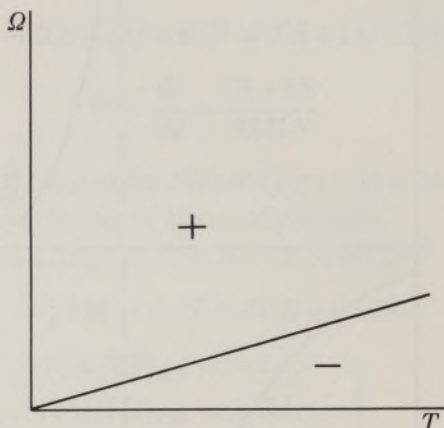


FIGURE 2. Phase diagram for Kerr black hole. The \pm phases correspond to positive and negative values of the specific heat at constant J . The equation of the dividing phase line is known explicitly as $\Omega = 0.23T$.

It is interesting to ask the physical significance of the discontinuity in $C_{J,Q}$. First note that at the critical value S , T , Ω and Φ all remain finite and continuous. In particular, the Gibbs free energy

$$G = M - TS - \Omega J - \Phi Q \quad (3.6)$$

is continuous, and has continuous gradients

$$(\partial G/\partial T)_{\Omega,\Phi}, \quad (\partial G/\partial \Omega)_{T,\Phi}, \quad (\partial G/\partial \Phi)_{T,\Omega}.$$

The phenomenon which occurs at the critical values of α and β may then be classified as a second order phase transition. Such a transition is characterized by continuity of G and its first derivatives, but a discontinuity in the second derivatives, e.g. heat capacity.

Figure 2 shows the phase diagram for this transition in the Kerr ($Q = 0$) case. The \pm denote the sign of the heat capacity. By eliminating M and S between (2.5), (2.6) and (2.7) one obtains for $J = 0.68M$

$$\Omega/T = \text{constant} \simeq 0.23 \quad (3.7)$$

so that the phase line in figure 2 is straight.

More surprisingly, in the Reissner–Nordstrom limit ($J = 0$), instead of (3.7) one obtains

$$\Phi = 1/\sqrt{3} \tag{3.8}$$

so that the phase transition takes place always at the same value of the electric potential, irrespective of the mass of the black hole.

Restricting attention to the Kerr ($Q = 0$) and Reissner–Nordstrom ($J = 0$) cases only, further information about the phase transition may be obtained by computing S as a function of Ω (or Φ) and T for the Kerr (Reissner–Nordstrom) cases respectively. After some work, one obtains

$$S = \frac{(1 - 16\Omega^2 S)^2}{2T^2(1 - 8\Omega^2 S)}, \tag{3.9}$$

and

$$S = \frac{(1 - \Phi^2)^2}{2T^2}. \tag{3.10}$$

We note in passing that, in terms of $\Theta = Q/2\Phi$ (3.10) reduces to the surprising result

$$S = \frac{1}{32\Theta^2} \tag{3.11}$$

which is independent of T .

It is now straightforward to compute more heat capacities

$$\begin{aligned} C_\Omega &\equiv T \left(\frac{\partial S}{\partial T} \right)_\Omega = - \left(\frac{1 - 16\Omega^2 S}{16\Omega^2 + T^2} \right) \\ &= - \frac{2TSM^{-1}}{(16\Omega^2 + T^2)}, \end{aligned} \tag{3.12}$$

and

$$C_\Phi \equiv T \left(\frac{\partial S}{\partial T} \right)_\Phi = -2S. \tag{3.13}$$

Moreover, we also readily obtain

$$\left(\frac{\partial J}{\partial T} \right)_\Omega = \left(\frac{\partial S}{\partial \Omega} \right)_T = - \frac{8M^3 \Omega S}{4M^3 \Omega^2 + TS} \tag{3.14}$$

$$\left(\frac{\partial Q}{\partial T} \right)_\Phi = \left(\frac{\partial S}{\partial \Phi} \right)_T = - \frac{1}{T}, \tag{3.15}$$

where the left hand equalities make use of the Maxwell relations between partial derivatives.

The quantities

$$\alpha = \begin{cases} - \frac{1}{J} \left(\frac{\partial J}{\partial T} \right)_\Omega \\ - \frac{1}{Q} \left(\frac{\partial Q}{\partial T} \right)_\Phi \end{cases}$$

which are obtained from (3.14) and (3.15) are analogous to the coefficient of thermal

expansion for ordinary matter. Here α may be envisaged as a coefficient of thermal rotation or electric charging.

Inspection of (3.12) and (3.13) shows that C_Ω and C_Φ are always negative, and are continuous across the phase line $\Omega = 0.23T$ or $\Phi = 1/\sqrt{3}$. The same is also true of α in both cases. Now, there is a well known differential identity leading to the relations

$$C_\Omega - C_J = TJ\alpha^2/K, \quad C_\Phi - C_Q = TQ\alpha^2/K, \tag{3.16}$$

where

$$K = \begin{cases} -\frac{1}{J} \left(\frac{\partial J}{\partial \Omega} \right)_T, \\ -\frac{1}{Q} \left(\frac{\partial Q}{\partial \Phi} \right)_T, \end{cases}$$

are the analogues of the isothermal compressibility, which may be envisaged here as the ease with which the black hole may be spun, or charged up, at constant temperature.

It may immediately be deduced from (3.16) and the continuity of C_Ω , C_Φ , α , J and Q that K must vanish at the phase transition, because C_Ω and C_Φ are infinite there. Indeed, K may be computed directly in the charged black hole case from the simple relation

$$Q = 2\Phi(1 - \Phi^2)/T \tag{3.17}$$

giving

$$K = -2(3\Phi^2 - 1)/TQ \tag{3.18}$$

which vanishes at $\Phi = 1/\sqrt{3}$ as expected.

It is possible to use the continuity of various quantities across the phase line to deduce analogues of the Clausius–Clapeyron relation for phase transitions. Continuity of J and Q gives

$$\left. \begin{array}{l} \frac{d\Omega}{dT} \\ \frac{d\Phi}{dT} \end{array} \right\} = \frac{\Delta\alpha}{\Delta K} \tag{3.19}$$

and continuity of S gives

$$\frac{d\Omega}{dT} = \frac{1}{JT} \frac{\Delta C_\Omega}{\Delta\alpha}, \quad \frac{d\Phi}{dT} = \frac{1}{QT} \frac{\Delta C_\Phi}{\Delta\alpha}, \tag{3.20}$$

where Δ denotes the jump in the following quantity, across the phase line. Knowing explicitly that $d\Omega/dT = 0.23$ and $d\Phi/dT = 0$ in this case (see equations (3.7) and (3.8)) confirms that α and K are continuous across the phase line, if C_Ω and C_Φ are.

On the other hand, by changing the variables from Ω or Φ to J or Q we obtain, instead of (3.20)

$$\frac{dJ}{dT} = \frac{\Delta C_J}{T\Delta(\partial\Omega/\partial T)_J}, \quad \frac{dQ}{dT} = \frac{\Delta C_Q}{T\Delta(\partial\Phi/\partial T)_Q}. \tag{3.21}$$

From equations (3.8) and (3.17) we have $dQ/dT = -4/(3\sqrt{3}T^2)$ and a similar but much more complicated expression reveals the continuity of dJ/dT . Hence we

deduce that $(\partial\Omega/\partial T)_J$ and $(\partial\Phi/\partial T)_Q$ also have infinite discontinuities at the phase transition. Direct computation, using the Maxwell relations, turns out to give

$$\left(\frac{\partial\Omega}{\partial T}\right)_J = \left(\frac{\partial S}{\partial J}\right)_T = \frac{16\Omega TS^3 + 2JS}{J^2 - 8T^2S^3}, \quad \left(\frac{\partial\Phi}{\partial T}\right)_Q = \left(\frac{\partial S}{\partial Q}\right)_T = \frac{QS}{\frac{3}{4}Q^2 - 2S}.$$

These expressions do indeed jump from $-\infty$ to $+\infty$ at the phase change (note that $S = \frac{9}{32}M^2$ and $Q^2 = \frac{3}{4}M^2$ at this point). The existence of a hitherto unsuspected

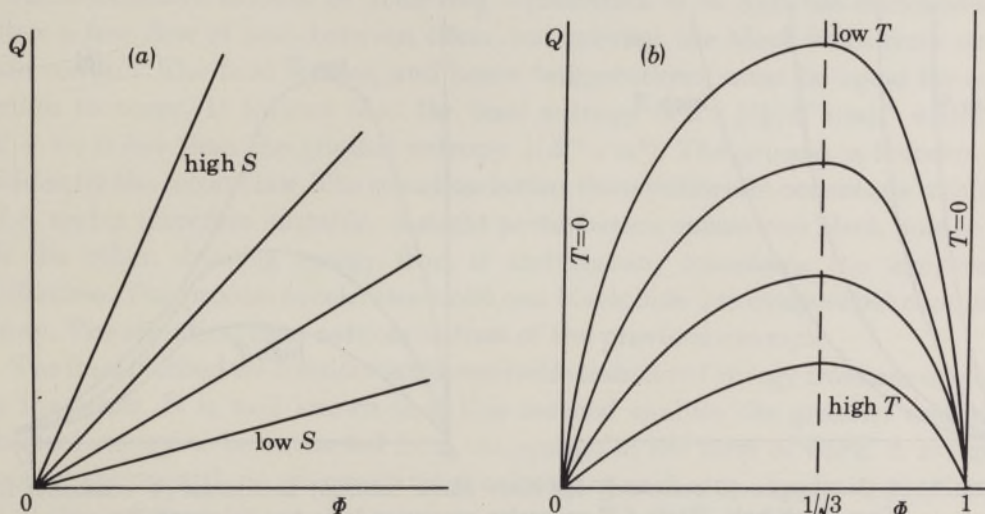


FIGURE 3. (a) Isentropes (S constant) for Reissner-Nordstrom black hole. (b) Isotherms (T constant) for Reissner-Nordstrom black hole. The broken line indicates the phase transition, which always occurs at $\Phi = 1/\sqrt{3}$. Each Q, T generally corresponds to *two* values of Φ . The point $\Phi = 1$ is the thermodynamic (black hole) limit, at which $T = 0$.

phase transition of black holes naturally raises the interesting question of whether other physical properties of black holes change at this point.

The ratio of rotational energy to 'gravitational' (i.e. surface or thermal) energy $\Omega J/TS$ in the Kerr case is given by $\alpha/[1 - \alpha + \sqrt{1 - \alpha}] \simeq 0.36$ at the critical point. Smarr (1973) has suggested that when this ratio is of order unity (which actually occurs at $\alpha^{\frac{1}{2}} = \frac{1}{2}\sqrt{3}$) then the Kerr solution might become unstable. Newtonian calculations (Bodenheimer & Ostriker 1973; Ostriker & Bodenheimer 1973; Ostriker & Peebles 1973) indicate that a rotating, self-gravitating fluid becomes unstable and loses its axisymmetry at a critical value of the ratio (rotational energy/Newtonian gravitational energy). I am grateful to Dr D. W. Sciama for drawing my attention to a striking result by Bertin & Radicati (1976), who have recovered this instability as a thermodynamic second order phase transition, at a value of the above ratio of about 0.14. In the higher angular momentum phase, the axisymmetric spheroid gives way to a triaxial ellipsoid. This suggests the possibility that in a similar fashion the Kerr black hole becomes unstable at $\alpha = 0.68$, and enters a non-axisymmetric, dynamic phase in which new thermodynamic degrees of freedom appear.

In figures 3*a* and 3*b* we give the general behaviour of the isotherms (T is a constant) and isentropes (S is a constant) on a Φ, Q diagram for a Reissner–Nordstrom black hole. In figure 4 the same quantities are plotted for the Kerr black hole on an Ω, J diagram. It is seen that for a given value of Q or J there are in general *two* possible values for Φ or Ω , giving the same temperature. One value corresponds to a small mass black hole with high Φ or Ω , the other to a large mass object with small Φ, Ω . At the phase transition point these two values coincide.

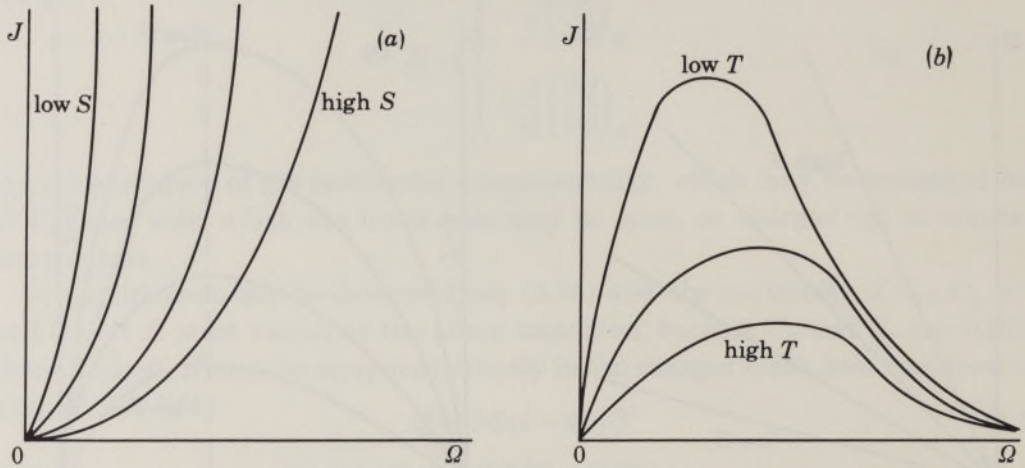


FIGURE 4. (a) Isentropes (S constant) for Kerr black hole. (b) Isotherms (T constant) for Kerr black hole. Each J, T generally corresponds to *two* values of Ω .

The utility of the ideas developed in this section will now be illustrated by applying them to some simple black hole processes. It is a general feature of the theory that, because the fundamental equation (2.3) is known precisely, quantities of interest may be computed directly from the various relations between the extensive and intensive parameters. A list of some useful such relations is given in the appendix. Usually in thermodynamics one only has available more limited information, e.g. one equation of state. It is then necessary to restrict attention to infinitesimal changes, dealing with the various partial derivatives such as α, K and C which can be experimentally measured, rather than directly with M, S, T , etc. Here the situation is the reverse.

As a first example we consider various processes involving two Schwarzschild black holes with masses M, m and temperatures T, t respectively. Each resides inside a small adiabatic enclosure, in equilibrium at constant temperature (see §4). The mass of the radiation will be ignored. Moreover, in order that each black hole may be assigned definite separate values of the thermodynamic parameters, it is necessary to assume that initially and finally the gravitational interaction between them is very small. This interaction can never be entirely eliminated, but provided the separation of the black holes greatly exceeds their Schwarzschild radii, then the thermodynamic disturbance from the equilibrium values assigned to each black hole in isolation is expected to be negligible.

There are various ways in which the two systems may be brought into equilibrium. One is to join the enclosures together and remove the dividing walls, thereby allowing the two black holes to coalesce adiabatically. The final mass is then the sum of the initial masses, $M + m$. The final temperature is $(M + m)^{-1} = Tt/(T + t)$ which is less than either T or t . The process is highly irreversible. The entropy jumps by $\frac{1}{2}Mm$ from $\frac{1}{4}(M^2 + m^2)$ to $\frac{1}{4}(M + m)^2$. Even if $T = t$, the entropy increase is non-zero, in contrast to a conventional thermodynamic system.

An alternative method of achieving equilibrium is to join the enclosures and allow a free flow of heat between them, but prevent the black holes from coming into contact. The final masses, and hence temperatures, must be equal for equilibrium to occur. It follows that the final entropy is $2 \times \frac{1}{4}(\frac{1}{2}(M + m))^2$ which, for $M \neq m$, is less than the original entropy $\frac{1}{4}(M^2 + m^2)$. The process is therefore forbidden by the second law. The situation is that the equilibrium occurs at a minimum of S , and is therefore unstable. A slight perturbation causes one black hole to feed on the other, drawing energy from it and thereby increasing the temperature difference. The process accelerates until one black hole has evaporated completely away. The situation then reduces to that of the previous example.

The final method we consider is the *reversible* transfer of energy from one enclosure to the other. It is well known that this method enables the greatest amount of thermal energy to be extracted from the system in the form of work. A reversible process occurs with no change of total entropy at each step. Applying the first law (2.5) this requires

$$dS = dM/T + dm/t = 0$$

where M, m, T, t are now considered as variables. Using (2.6) gives

$$M dM + m dm = 0,$$

and integrating this yields $M_f = [\frac{1}{2}(M_i^2 + m_i^2)]^{\frac{1}{2}}$

$$T_f = \frac{\sqrt{2T_1 t_1}}{(T_1^2 + t_1^2)^{\frac{1}{2}}}$$

for the final mass and temperature of each black hole in terms of the initial values M_1, t_1 , etc. The energy extracted during the process is

$$M_1 + m_1 - 2[\frac{1}{2}(M_1^2 + m_1^2)]^{\frac{1}{2}}.$$

Note that the final temperature is the maximum equilibrium temperature that can be achieved.

If $J, Q \neq 0$ the range of possible processes which can be treated by these methods is enormous. For example, if coalescence is carried out reversibly (somehow!) rather than adiabatically, then the maximum energy may be extracted. Putting S to be constant, we note from equation (2.3) that the final mass will be lowest when J is a minimum. This implies that maximum energy extraction will occur if the two rotating black holes are coalesced along their common rotation axis when counter-rotating.

As a final illustration, consider two charged, non-rotating black holes in separate adiabatic enclosures. For simplicity assume that they remain at a fixed common temperature T , but have different initial charges Q_1, q_1 and masses M_1, m_1 . They are allowed to come reversibly into mechanical (electrical) equilibrium, with maximum work being delivered to the outside. We attempt to answer the question: what are the final masses?

Applying the first law (2.5) with $dS = 0$ yields

$$dM + dm - \Phi dQ - \phi dq = 0.$$

Noting the relation (3.17) and

$$MT = 1 - \Phi^4$$

leads to

$$(\Phi^3 - \Phi) d\Phi + (\phi^3 - \phi) d\phi = 0. \quad (3.22)$$

Electrical equilibrium occurs when $\Phi = \phi = \Phi_f$, say, which, for $T = \text{constant}$, implies from (3.21) that $M = m$. Integrating (3.22) from Φ_1 and ϕ_1 to Φ_f gives, after substituting for Φ_s

$$2(1 - M_f T)^2 (M_f^2 T^2 - 2M_f T - 1) = (1 - M_1 T)^2 (M_1^2 T^2 - 2M_1 T - 1) \\ + (1 - M_1 T)^2 (m_1 T^2 - 2m_1 T - 1)$$

for the final mass M_f in terms of the initial masses M_1, m_1 , or alternatively of the initial charges if (3.17) is used. The energy extracted is $2M_f - (M_1 + m_1)$.

Various processes may also be carried out on single black holes. For example, they may be charged up isothermally. In this case the energy required is obtained directly from relations (3.21) and (3.16) for a given change in Q or Φ . Note that for a small change

$$\left(\frac{\partial M}{\partial \Phi}\right)_T = -\frac{4\Phi^3}{T} = TQ\alpha + \Phi QK.$$

Analogous processes occur for rotating black holes utilizing the Penrose mechanism for changing the rotational energy. They too may be studied by the techniques given here.

4. STABILITY AND EQUILIBRIUM CONDITIONS

One of the central achievements of thermodynamics is its ability to predict the conditions for stable equilibrium, even for multicomponent systems, by using the second law. This treatment may also be successfully applied to black holes in contact with heat reservoirs, energy sources, etc.

Inspection of equation (2.4) reveals that, for a given mass black hole, S is maximized by choosing $J = Q = 0$. This suggests that, left to itself, a black hole will spin down and discharge. This is indeed the case. The phenomenon of superradiance (Unruh 1974) predicts that boson quanta of axial angular momentum m and frequency $\omega < m\Omega$ cause stimulated emission of rotational energy. As originally suggested by Zel'dovich (1970), there is also spontaneous spin-down by emission

into these modes. An analogous process occurs for charged black holes (Gibbons 1975) for modes with $\omega < e\Phi$ where e is the charge on the emitted quantum. In the former case the spontaneous spin-down rate is very slow, except for microscopic black holes. On the other hand the electric discharge process is very much more efficient. In order to apply the ideas of thermodynamics it is necessary to consider processes whose characteristic time scale is much less than these decay times. For supermassive black holes ($M \geq 10^5 M_0$) the electric discharge will cease completely, because insufficient thermal and electromagnetic energy is available to cause the emission of charged particles with a non-zero rest mass (see Carter 1974). In this regime, the thermodynamic approximation is fully justified for non-rotating black holes.

If a black hole is immersed in a bath of thermal radiation at some temperature T , it is possible for the non-superradiant modes to come into equilibrium. If the effects of superradiance are neglected, we may treat J and Q for the black hole as constants. The condition for equilibrium under these circumstances is that the Helmholtz free energy F be at a turning point. If in addition it is a minimum then the equilibrium will be stable. F is here defined as

$$F = M - TS$$

so we require $dF = 0$, and $d^2F > 0$ for stability. The condition $dF = 0$ merely reproduces the relation between M , T and S which follows from (2.6)–(2.8). This shows that a black hole of any M , J and Q may be in equilibrium with thermal radiation, provided that the temperature T is the same. However, application of the stability criterion gives, on performing a second differentiation of F

$$2(1 - \alpha - \beta)^{\frac{3}{2}} < 6\alpha - \alpha\beta - 3\beta - 2.$$

Squaring both sides of this equation and factoring yields

$$(4\alpha + \beta^2)(\alpha^2 + 6\alpha + 4\beta - 3) > 0$$

from which it follows that $\alpha^2 + 6\alpha + 4\beta - 3 > 0$. Inspection of (3.3) reveals that this is just the condition that the black hole be in the higher J, Q phase. This is the expected result because in this phase the black hole heat capacity is positive. For values of α and β below the critical transition values, the equilibrium is unstable. The negative heat capacity in this regime implies that a slight drop in black hole temperature will cause a further drop as energy is absorbed. The process will continue indefinitely, with the black hole feeding on the surrounding heat bath. Conversely, a slight rise in black hole temperature from the equilibrium value will cause the black hole to radiate some net energy, thereby increasing its temperature still further, eventually leading to the explosive disappearance of the black hole altogether.

Adiabatic equilibrium may also be treated by similar techniques. If a black hole is placed in a box with perfectly reflecting walls, account must be taken of the accumulation of thermal evaporation radiation inside the box. Under these

circumstances, the equilibrium condition is that the total entropy of the box contents be an extremum. Stability occurs if the extremum is a maximum. For simplicity we treat only the Schwarzschild ($J = Q = 0$) case.

The total entropy of black hole + radiation is

$$S = \frac{1}{4}M^2 + \frac{4}{3}(aVM_r^3)^{\frac{1}{4}} \quad (4.1)$$

where M_r denotes the mass of the radiation, V is the volume of the box and a is the radiation constant. The condition for equilibrium is that $dS = 0$ subject to the adiabatic constraint $dM + dM_r = 0$. The result is

$$V = M_r M^4/a. \quad (4.2)$$

For stability, it is necessary to have $d^2S < 0$. Differentiating (4.1) a second time and using (4.2) to eliminate aV yields

$$M > 4M_r \quad (4.3)$$

or
$$V < \frac{1}{4}M^5/a. \quad (4.4)$$

It follows that Schwarzschild black holes with masses less than four fifths of the total cannot be in stable adiabatic equilibrium with thermal radiation. They will either evaporate away, or will grow in mass until conditions (4.3) and (4.4) are satisfied. (This result was first pointed out by Hawking (1976).)

In the case that $J, Q \neq 0$, the maximum volume of the box for stability increases until, at the critical values (3.3), V is infinite. If we require isothermal or adiabatic equilibrium with Ω and Φ held constant, it is necessary to deal instead with the Gibbs function G or the enthalpy, respectively.

If a black hole is to form in the first place, it might be expected that its entropy should be greater than that of its initial constituents. In the case of a Schwarzschild black hole formed out of thermal radiation this requires

$$\frac{1}{4}M^2 > \frac{4}{3}(aVM^3)^{\frac{1}{4}} \quad (4.5)$$

or
$$V < \left(\frac{3}{16}\right)^4 M^5/a \quad (4.6)$$

which should be compared with (4.4). For a radiation temperature T_r condition (4.6) implies

$$MT_r > \frac{16}{3}. \quad (4.7)$$

This restriction might place a limit on the size of microscopic black holes formed in the radiation dominated era of the primeval fireball. At that epoch, the average temperature of the cosmological fluid is about $10^{10} \text{K}/t^{\frac{1}{2}}$ seconds. Condition (4.7) gives

$$M > \text{constant} \times t^{\frac{1}{2}}. \quad (4.8)$$

For t in seconds and M in grams, the value of the constant turns out to be about 10^{15} . On the other hand Carr & Hawking (1974) find from a study of equations of state of primordial matter that $M > 10^{38}t$ in the same units. Clearly the restriction (4.8) is quite irrelevant.

On the other hand, for a black hole formed out of non-relativistic matter, condition (4.5) must be replaced by the approximate inequality

$$M^2 \geq N = M/m \tag{4.9}$$

where N are the number of particles of mass m . In the case of protons (4.9) gives

$$M \geq 10^{15} \text{ g.}$$

This limit makes it appear unlikely that microscopic black holes ever form from non-relativistic matter.

5. NAKED SINGULARITIES AND THE THIRD LAW

In its original form, the third law of thermodynamics due to Nernst states that the entropy differences between states that can be connected by an isothermal process must vanish as $T \rightarrow 0$. Sometimes this is interpreted as implying the unattainability of the absolute zero of temperature. If the latter is applied to black holes (see Carter 1973) then, using equation (2.6) we obtain, for $T \rightarrow 0$

$$J^2 + \frac{1}{4}Q^2 < 16S$$

for a given mass M . From equation (2.4) it then follows that

$$\frac{J^2}{M^4} + \frac{Q^2}{M^2} < 1$$

or $\alpha + \beta < 1$.

This is the condition that the Kerr–Newmann family of solutions describe black holes and not naked singularities. Hence the unattainability of absolute zero is equivalent to the cosmic censorship hypothesis (Penrose 1969). It is therefore of some interest to examine how well the Nernst postulate holds up for black holes.

Inspection of equations (3.14) and (3.15) reveals that

$$\left(\frac{\partial S}{\partial \Omega}\right)_T \rightarrow -\frac{2S}{\Omega} = -M^3 \text{ as } T \rightarrow 0 \tag{5.1}$$

$$\left(\frac{\partial S}{\partial \Phi}\right)_T \rightarrow -\frac{2M}{T} \rightarrow \infty \text{ as } T \rightarrow 0. \tag{5.2}$$

Thus, the isothermal quantity α does *not* vanish at $T = 0$, in apparent contradiction to the Nernst postulate. The same is also true of other entropy-change quantities, e.g. $(\partial S/\partial Q)_T$.

The third law is in fact usually accepted in a stronger form due to Planck. This states that the entropy of a quantum thermodynamic system *vanishes* as $T \rightarrow 0$. Inspection of (3.2) shows that

$$C_{J,Q} \rightarrow 0 \text{ as } T \rightarrow 0$$

so that the entropy change in a constant J, Q process certainly remains finite as $T = 0$ is approached

$$\Delta S = \int_0^T \frac{C_{J,Q}}{T} dT < \infty.$$

This remains true for a constant Ω process. From (3.12)

$$C_\Omega \rightarrow 0 \text{ as } T \rightarrow 0.$$

Nevertheless, it is clear from (2.4) that

$$S \rightarrow \frac{1}{4}M^2 - \frac{1}{8}Q^2 > 0 \text{ as } T \rightarrow 0.$$

It is not possible to remove this zero point entropy with an additive constant, because M and Q are extensive parameters. Moreover, the statistical foundation of

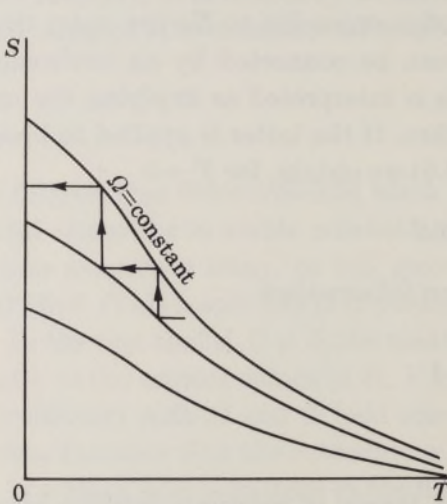


FIGURE 5

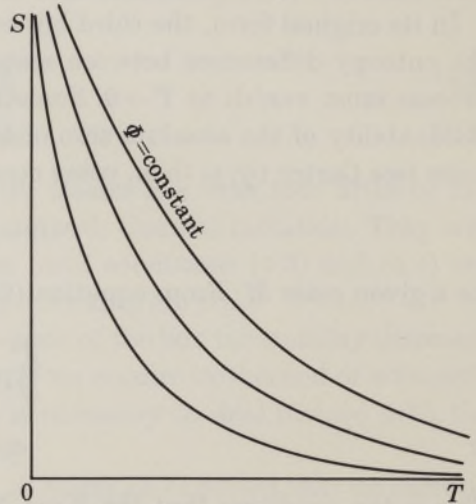


FIGURE 6

FIGURE 5. Violation of the third law? The entropy of a Kerr black hole does not vanish at $T = 0$, so lines of constant Ω do not intersect there. A thermodynamically permitted path which reaches absolute zero in a finite sequence of transitions is shown.

FIGURE 6. Lines of constant $\Phi < 1$ do not intersect $T = 0$.

Planck's law is that there exists only one, unique, microstate associated with the zero temperature state. There is a consistent statistical basis for the black hole entropy which is based on an examination of the microscopic particle states which may make up a black hole with given M, J and Q (Christodoulou & Ruffini 1971; Bekenstein 1975). Clearly an extreme Kerr–Newman black hole of given mass may be constructed from many different microscopic configurations.

Because there exists an M -dependent zero point entropy, lines of constant Ω (with $Q = 0$) do not intersect at $T = 0$ on an S – T diagram (see figure 5). This situation is usually associated with a violation of the third law, because a reversible process (shown as arrows) may, in a finite number of steps, be used to cool the system to $T = 0$. However, caution should be observed in concluding that there exist thermodynamic ways of converting a black hole into a naked singularity.

There may be reasons (perhaps non-thermodynamic) why the process of interest cannot occur. For example, although C_Φ remains finite at $T = 0$ (see (3.13)) in apparent violation of the third law, it is not possible to cool a black hole along a line of constant Φ to $T = 0$. The reason for this may be seen by eliminating S from (2.8) to obtain

$$\frac{2\Phi}{1 + \Phi^2} = \frac{Q}{M} = \beta^{\frac{1}{2}}$$

from which it follows that for a given Φ , β is constant. Hence it is not possible to approach $T = 0$, $\beta = 1$ with fixed Φ . (See figure 6.)

It is interesting to determine what would happen if a black hole were cooled into a naked singularity. For a non-rotating supermassive object carrying an enormous electric charge, the disturbances due to superradiance disappear, and the black hole may be in complete stable equilibrium with its environment with constant charge. If $Q/M > \frac{1}{2}\sqrt{3}$, C_Q is positive, so the black hole may be cooled down in the traditional way by manipulating the environment and allowing the black hole to radiate energy. Whether or not M may be decreased until $Q/M = 1$ ($T = 0$) depends on whether the third law will turn out to apply to black holes after all. But if this happens, a slight disturbance will then convert the black hole into a naked singularity. We are then faced with the fascinating question: what happens? Does a naked singularity radiate also? Will the collapsing matter radiate away before the singularity forms, thus removing some of the repulsiveness of these entities?

Fortunately, the techniques developed by the author with S. A. Fulling & W. G. Unruh for studying the energy-momentum tensor of a quantum field in the vicinity of a black hole (Davies, Fulling & Unruh 1975; Davies 1976*a*; Davies & Fulling 1977) may be immediately extended to deal with the naked singularity case. The emission of radiation from a Reissner-Nordstrom black hole, or spherically symmetric collapse to a naked singularity, may be treated by using a two-dimensional model. Such a model has been successfully applied to the calculation of the vacuum expectation value of a massless scalar field in the vicinity of an electrically neutral black hole, for a general collapse scenario. The detailed treatment, which is readily extended to the charged case, has been given in Davies (1976*a*). Consequently, only an outline of the extension will be given here.

The exterior and interior metrics of the model star are taken to be

$$ds^2 = (1 - 2M/r + Q^2/r^2) du dv, \quad r > R(\tau), \tag{5.3}$$

$$ds^2 = A(U, V) dU dV, \quad r < R(\tau), \tag{5.4}$$

respectively, where

$$U = \tau - r + R_0, \quad V = \tau + r - R_0,$$

$$u = t - r^* + R_0^*, \quad v = t + r^* - R_0^*,$$

$$r^* = \int \frac{dr}{1 - 2M/r + Q^2/r^2}.$$

At time $t = 0$, the surface of the star collapses inwards along the trajectory $r = R(\tau)$.

According to the general theory, the expectation value of the energy-momentum tensor of a massless scalar field in the background metric of the star, evaluated in the initial ($t < 0$) quantum vacuum is given by

$$\left. \begin{aligned} T_{\bar{u}\bar{u}} &= -F_{\bar{u}}(C), \\ T_{\bar{v}\bar{v}} &= -F_{\bar{v}}(C), \end{aligned} \right\} \quad (5.5)$$

where F denotes the functional form

$$F_x(y) = (12\pi)^{-1} y^{\frac{1}{2}} \frac{\partial^2}{\partial x^2} (y^{-\frac{1}{2}}).$$

The coordinates \bar{u} and \bar{v} are defined in terms of the transformation functions $\alpha(u) = U$ and $\beta(V) = v$ through

$$\begin{aligned} \bar{v} &= v = \beta(V), \\ \bar{u} &= \beta[\alpha(u) - 2R_0] = \beta[U - 2R_0]. \end{aligned}$$

Only derivatives of α and β are needed. They are defined through equation (5.7) below. The quantity C in (5.5) is the conformal factor in the metric expressed in terms of the \bar{u}, \bar{v} coordinates:

$$C = \left\{ \begin{aligned} &A(U, V) [\beta'(U - 2R_0)]^{-1} [\beta'(V)]^{-1}, \quad r < R(\tau) \\ &((1 - 2M/r + Q^2/r^2) [\alpha'(u)]^{-1} [\beta'(U - 2R_0)]^{-1}), \quad r > R(\tau). \end{aligned} \right\} \quad (5.6)$$

The expressions for $\alpha'(u) = dU/du$ and $[\beta'(V)]^{-1} = dV/dv$ are

$$\left(1 - \frac{2M}{R} + \frac{Q^2}{R^2}\right) (1 \mp \dot{R}) \left[A \left(1 - \frac{2M}{R} + \frac{Q^2}{R^2} + \frac{2M\dot{R}^2}{R} - \frac{Q^2\dot{R}^2}{R^2}\right) + (1 - A)\dot{R}^2 \right]^{\frac{1}{2}} \mp \dot{R} \quad (5.7)$$

(\pm respectively) where $A = A(U, V)$, $U = \alpha(u) = \tau - R(\tau) + R_0$, $V = \tau + R(\tau) - R_0$, $R = R(\tau)$ and \dot{R} denotes $dR/d\tau$.

As we are here only interested in the situation outside the collapsing star, substitute the exterior metric from (5.6) into (5.5) to obtain, in u, v coordinates

$$\left. \begin{aligned} T_{uu} &= \alpha'^2 F_U(\beta') + F_u(\alpha') + (\text{static term}), \\ T_{vv} &= (\text{static term}), \end{aligned} \right\} \quad (5.8)$$

where

$$(\text{static term}) = (12\pi)^{-1} \left[\frac{3M^2}{4r^4} - \frac{M}{2r^3} + \frac{Q^4}{2r^6} + \frac{3Q^2}{4r^4} - \frac{3Q^2M}{2r^5} \right]. \quad (5.9)$$

As explained in Davies (1976*a*) the first term on the right of (5.8) represents radiation from the far side of the star coming through the centre. The static term represents vacuum polarization induced by the spacetime curvature near the star. It is always negative outside the event horizon in the black hole case. The term $F_u(\alpha')$ represents a flux of radiation coming out of the collapsing object. If a black hole forms, this flux settles down to a value independent of the details of the collapse. Its value is

$$\frac{1}{48\pi} \left(\frac{M}{r_+^2} - \frac{Q^2}{r_+^3} \right). \quad (5.10)$$

where $r_+ = M + (M^2 - Q^2)^{\frac{1}{2}}$ is the radius of the event horizon. This is the only contribution to the energy-momentum tensor which is important for a distant observer ($r \rightarrow \infty$). The $\alpha'^2 F_U(\beta')$ term fades out exponentially, while the vacuum polarization falls to zero like r^{-3} . The term (5.10) represents the thermal emission from the black hole discovered by Hawking (1975). It clearly vanishes when $Q = M$, and reduces to the value given in Davies, Fulling & Unruh when $Q = 0$.

In two dimensions, the radiation flux from a blackbody at temperature T is

$$\frac{1}{2\pi} \int_0^\infty \frac{\omega d\omega}{\exp(\omega/kT) - 1} = \frac{\pi k^2 T^2}{12}. \tag{5.11}$$

Choosing our units $k = (8\pi)^{-1}$ and equating (4.17) and (4.18) gives

$$T = 4 \left(\frac{M}{r_+^2} - \frac{Q^2}{r_+^3} \right). \tag{5.12}$$

Using equation (2.3) enables us to write $r_+ = 4M^{-1}(S + \frac{1}{8}Q^2)$ so (5.12) becomes

$$\begin{aligned} T &= \frac{M^3(S - \frac{1}{8}Q^2)}{4(S + \frac{1}{8}Q^2)^3} \\ &= M^{-1}(1 - Q^4/64S^2). \end{aligned}$$

This result is in complete agreement with the thermodynamic definition (2.6) which follows from the generalized Smarr formula. It confirms the underlying consistency of the energy-momentum tensor calculations.

The formation of the event horizon allows the ingoing flux T_{vv} to stream across the event horizon into the black hole. By using the definition of r_+ it is possible, after some algebra, to show that the value of the static term, and hence T_{vv} , at the horizon is

$$-\frac{1}{48\pi} \left(\frac{M}{r_+^2} - \frac{Q^2}{r_+^3} \right)^2.$$

This negative flux into the black hole reduces the mass of the black hole at the right rate to compensate for the thermal flux at infinity. Moreover, it follows that the last two terms of T_{uu} cancel at $r = r_+$. This is important because it means that an observer who falls into the black hole is not burnt up by an intensely blue-shifted flux of Hawking radiation, as is sometimes suggested. The reason for this is that, for a falling observer, all the terms in equation (5.8) are of comparable importance. The first term, which fades out exponentially for an observer at infinity, remains finite when measured by the falling observer crossing the horizon. The vacuum polarization, which is a static cloud of negative energy density in the Schwarzschild (r, t) frame, gives rise to a negative flux of energy as the falling observer sweeps through it. Although this contribution becomes infinite at the horizon, it is compensated by the thermal flux which is also becoming infinite, but positive, as the observer approaches the horizon. In this way, the total measured flux at $r = r_+$ remains finite. This may be exhibited explicitly by transforming $T_{\mu\nu}$ into Kruskal-type coordinates. It also happens that $T_{\mu\nu}$ remains finite at the inner surface

$r = r_- = M - (M^2 - Q^2)^{\frac{1}{2}}$. An observer who passes through this surface can travel to another region of spacetime connected through the black hole. It is often conjectured that in the real universe, the infalling observer would encounter a singularity along the surface $r = r_-$ due to infinite blue shift effects. However, such effects evidently do not occur from the Hawking flux and the vacuum polarization. The falling observer may still be vaporized by the radiation which falls in after him, but he will not be vaporized by the heat of the black hole.

We now turn to the case $Q > M$, for which the infalling matter classically encounters a naked singularity. What effect does a quantum treatment have? The energy-momentum tensor is still given by equations (5.8) and (5.9), but in the absence of an event horizon the $F_u(\alpha')$ term does not approach the value (5.10), and the $\alpha'^2 F_U(\beta')$ term no longer becomes negligible for an observer at infinity. Instead T_{uu} is given by a long and complicated function of A , $R(\tau)$ and its derivatives.

However, as a preliminary investigation, we give a treatment of the special case of a uniformly collapsing thin shell, for which $A = 1$ and $R(\tau) = R_0 - \nu\tau$. In the case of rapid collapse, $\nu \simeq 1$. This case was treated by Davies *et al.* (1976) for uncharged black holes. The treatment reported here follows along the lines of that paper. Provided R is not approaching zero, one obtains for T_{uu}

$$-\frac{1}{12\pi} \left[\frac{3M^2}{4R^4} - \frac{M}{2R^3} + \frac{Q^4}{2R^6} + \frac{3Q^2}{4R^4} - \frac{3Q^2M}{2R^5} \right]. \quad (5.13)$$

This expression is now valid for any Q and M . For $Q < M$ it agrees with (5.10) at $R = r_+$. As $R \rightarrow 0$ all the terms rise in value without limit, but the dominant contribution comes from the *electromagnetic* terms (even if the charge Q is small). At small R we have to be careful about the limit $\nu \simeq 1$. A laborious exact calculation reveals that, for any $\nu < 1$ and $R \rightarrow 0$

$$T_{uu} \rightarrow \frac{\nu^3}{3\pi(1+\nu)(1-\nu)^3} \frac{Q^2}{R^4}. \quad (5.14)$$

For $Q < M$ this diverging flux stays inside the black hole. However, for $Q > M$ there is no horizon. We arrive at the important conclusion that matter collapsing towards a classical naked singularity also radiates energy when quantum effects are considered. The spectrum is not thermal (so thermodynamic techniques do not apply as the situation is manifestly not an equilibrium one) and the intensity rises without limit as the singularity is approached. In a real collapse, back-reaction would occur and reduce the mass of the collapsing matter. From the form of (5.14) we see that the integrated outgoing flux approaches M as R approaches $M^{\frac{1}{2}}$ (assuming $Q \approx M$). So for collapsing objects with masses well above the Planck mass, all the mass will be radiated away† long before the matter shrinks inside the Planck length and quantum gravity becomes important. This suggests that purely quantum matter effects will prevent the formation of naked singularities in this case. In the case of cooling a black hole down below $T = 0$ (i.e. starting with $Q < M$ and allowing

† As there is no horizon, baryon number will be conserved.

M to fall through quantum radiation effects) the situation is less clear. It could be that a horizon will form around the late stages of collapse, thus preventing the radiation from escaping, and the back-reaction may not then disrupt the collapse. The details of this process must await a full understanding of the mechanism of back-reaction on the matter.

6. OUTLOOK AND CONCLUSIONS

The recent developments on the quantum properties of black holes make it likely that they are subject at least to the first and second laws of thermodynamics. The formal assignment of an entropy to a black hole was originally due to an examination of the microscopic particle states which can go to make up the black hole (Christodoulou & Ruffini 1971; Bekenstein 1972). In conventional statistical mechanics, the entropy of a thermodynamic macrostate may be defined as

$$S = k \ln W, \quad (6.1)$$

where W is the statistical weight of the macrostate with entropy S , i.e. the number of possible microstates consistent with it.

However, a definition based on (6.1) for a black hole is really only formal because there is no way, even in principle, in which we could observe the constituent microstates from outside the black hole. Moreover, as pointed out in §2, it is not possible to consider the black hole entropy as being located with a certain density in a certain region of spacetime. Black hole entropy is a truly global property.

These considerations suggest that the entropy is more a property of the gravitational field than the microscopic matter content of the black hole. It could, after all, be made entirely of gravitons. Moreover, the temperature, and hence the entropy of a black hole are certainly affected by the distribution of matter around them. For example, the temperature of radiation from a black hole inside a static shell of matter is lower than one in free space. The contribution of such gravitational fields to the entropy should be taken into account (Davies 1976*b*) so that the tendency of self-gravitating systems to grow more lumpy with time, and for space-time curvature to increase, may be given a thermodynamical basis (see also Penrose 1974). If the black hole entropy is really gravitational in origin, we might expect to encounter entropy even in the absence of matter. Empty universes with gravitational fields can occur if Einstein's cosmological constant is non-zero. In this case one has horizons of the de Sitter type. Recently, Gibbons & Hawking (1976) have argued that there is indeed an entropy associated with de Sitter horizons, even though there is no matter beyond them.

Future work should be directed to understanding the full physical significance of the phase transition discussed in §3, and attempting to extend the thermodynamic description to non-equilibrium situations. It might then be possible to treat self-gravitating systems other than black holes, and hence to ascertain a more comprehensive relation between entropy and gravity.

I should like to thank Dr Bernard Carr, Dr Stephen Fulling, Dr Dennis Sciama, and Professor Roger Penrose, F.R.S., for helpful comments and discussions. Professor Peter Landsberg has kindly drawn my attention to an independent alternative derivation by him of the violation of the third law of thermodynamics by black holes (private communication).

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APPENDIX

The following relations are often found useful in calculations.

Kerr

$$2S = M^2(1 - 8\Omega^2 S)$$

$$S = \frac{(1 - 16\Omega^2 S)^2}{2T^2(1 - 8\Omega^2 S)}$$

$$MT = 2 - M^2/2S$$

$$T^{-2}(1 - J^2/16S^2)^2 = 2S + J^2/8S$$

$$M^2 = 2S(2 - MT)$$

$$J^2 = 16S^2(1 - MT)$$

$$2TS/M = 1 - 16\Omega^2 S$$

$$MT = 1 - 4M^2\Omega^2$$

$$\Omega^2 = \frac{J^2}{8S(16S^2 + J^2)}$$

Reissner-Nordstrom

$$Q^2 = 8S\Phi^2$$

$$Q = 2\Phi$$

$$M = 1 + \Phi^2$$

$$MT = 1 - \Phi^4$$

$$Q = 2\Phi(1 - \Phi^2)/T$$