

II. Inflation and the cosmological perturbations

1. Introduction:

The Origin of structure in the universe is originated from small 'seed' perturbations

Once the universe becomes matter dominated (~ 1000 yrs) after the bang), primeval density inhomogeneities ($\delta\rho/\rho \sim 10^{-5}$) are amplified by gravity and grow into the structure we see today.

Proof: COBE discovery of CMB anisotropies

The temperature anisotropies detected almost certainly owe their existence to primeval density inhomogeneities.

Explanation of the small seed perturbations?

Our best guess of the origin of these perturbations is quantum fluctuations during an inflationary era.

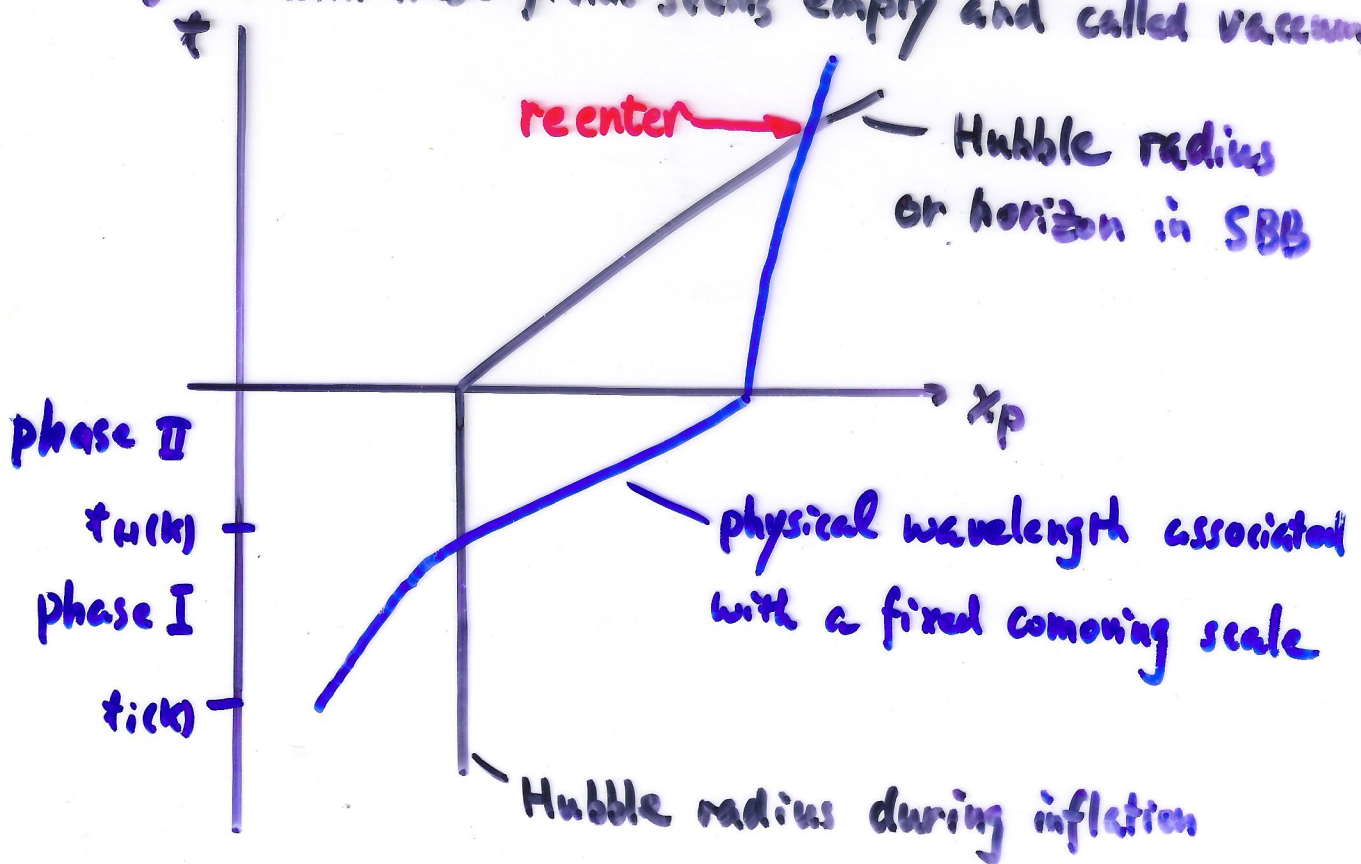
Inflation { horizon
flatness problems
entropy

generate spectra of both density perturbations and gravitational waves. — most useful property of inflation

Vacuum: QFT: empty space is not entirely empty
 it is filled with quantum fluctuations of all
 types of physical fields.

The fluctuations can be regarded as waves of
 physical fields with all possible wavelengths, moving
 in all directions.

If the values of these fields, averaged over some
 macroscopically long time, vanish then the space
 filled with these fields seems empty and called vacuum.



Phase I: The wavelength of all vacuum fluctuations of the
 inflaton field grow exponentially

Phase II: Wavelength of any fluctuation $> H^{-1}$, fluctuation stops
 propagating, its amplitude freezes at some non-zero $\delta\phi$

Reenter: The spectra provides a distinctive signature of inflation

Quantum fluctuations of the inflaton field are generated during inflation.

Gravity talks to any component of the universe

Small fluctuations of the inflaton field

fluctuations of the spacetime metric \rightarrow perturbation of the curvature R

The evolution of Curvature perturbation:

Subhorizon: it is generated, grow

superhorizon: frozen and may be considered as classical

reenter: curvature perturbations of the spacetime give rise to matter perturbations $\delta\rho$. These fluctuations will then grow to give rise to the structure today.

Two are the key ingredients for understanding the observed structures in the universe with inflation.

1. Quantum fluctuations of the inflaton field are excited during inflation and stretched to cosmological scales. Meanwhile inflaton fluctuations connecting to the metric perturbations, thus the metric is also excited and stretched to cosmological scales.
2. Once a given wavelength becomes smaller than the horizon scale after inflation, gravity acts as a messenger and communicates to baryons and photons the small seed perturbation.

2. Quantum fluctuations of a generic massless scalar field during inflation

1) During a de Sitter stage

Expanding the scalar field χ in Fourier modes

$$\delta\chi(x, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \delta\chi_{\vec{k}}(t)$$

We can write the equation for the fluctuation as

$$\delta\ddot{\chi}_{\vec{k}} + 3H\delta\dot{\chi}_{\vec{k}} + \frac{k^2}{a^2}\delta\chi_{\vec{k}} = 0$$

Let's first study the qualitative behavior of $\delta\chi_{\vec{k}}$.

a. For wavelength within the horizon $\lambda = \frac{a}{k} \ll H^{-1}$, the corresponding wavenumber $k \gg aH$. In this regime, we can neglect the friction term $3H\delta\dot{\chi}_{\vec{k}}$, thus

$$\delta\ddot{\chi}_{\vec{k}} + \frac{k^2}{a^2}\delta\chi_{\vec{k}} = 0 \quad \delta\chi_{\vec{k}} \text{ oscillates}$$

b. For wavelength above the horizon $\lambda = \frac{a}{k} \gg H^{-1}$, the wave number $k \ll aH$ and the term k^2/a^2 can be neglected

$$\delta\ddot{\chi}_{\vec{k}} + 3H\delta\dot{\chi}_{\vec{k}} = 0$$

$\delta\chi_{\vec{k}}$ remains constant on superhorizon scale

We have the following picture: Take a given fluctuation whose initial wavelength $\lambda \sim a/k$ is within the horizon, the fluctuations oscillates till the wavelength becomes of the order of the horizon scale. When the wavelength crosses the horizon, the fluctuation ceases to oscillate and gets frozen in.

Quantitative study the fluctuation evolution:

perform the redefinition

$$\delta \chi_k = \frac{\delta \sigma_k}{a}$$

and we work in conformal time $d\tau = dt/a$. Then

$$a \sim e^{H\tau} \Rightarrow a(\tau) = -\frac{1}{H\tau} \quad (\tau < 0)$$

Equation of motion for $\delta \chi_k \rightarrow \delta \sigma_k$

$$\delta \sigma_k'' + \left(k^2 - \frac{\ddot{a}}{a}\right) \delta \sigma_k = 0$$

This is very close to Klein-Gordon Eq. for scalar field in flat spacetime, the only difference is the minus sign.

On subhorizon scales: $\lambda_{\text{ph}} = \frac{a}{k} \ll H^{-1} \Rightarrow -k\tau \gg 1$,

$$\delta \sigma_k'' + k^2 \delta \sigma_k = 0$$

whose solution is a plane wave

$$\delta \sigma_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \quad (k \gg aH)$$

On superhorizon $\lambda_{pH} = \frac{a}{k} \gg H^{-1} \Rightarrow -k\tau \ll -1$

$$\delta\sigma_k'' - \frac{a''}{a}\delta\sigma_k = 0$$

which is satisfied by $\delta\sigma_k = B(k)a$

where $B(k)$ is a constant of integration. At the horizon crossing ($k = aH$, or $-k\tau = 1$), matching the solutions of sub- & super-horizons

$$a B(k) = \frac{e^{-ik\tau}}{\sqrt{2k}} \Big|_{k\tau=1} \Rightarrow B(k) = \frac{e^{-ik\tau}}{\sqrt{2k}} \frac{1}{a} \Big|_{\frac{k}{a}=H} = \frac{e^{-ik\tau}}{\sqrt{2k}} \frac{H}{k}$$

$$\stackrel{k \gg 0}{\sim} \frac{H}{\sqrt{2k^{3/2}}}$$

Going back to the original $\delta\chi_k$, we obtain that the quantum fluctuation of the χ field on superhorizon scales is a constant and approximately equal to

$$|\delta\chi_k| \approx \frac{H}{\sqrt{2k^3}}$$

2) Massive scalar field quantum fluctuations during a de Sitter stage

Let's discuss the solution when a mass term presents

$$\delta\sigma_k'' + [k^2 + M^2(\tau)]\delta\sigma_k = 0$$

where

$$M^2(\tau) = (m^2 - 2H^2)\tau^2 = \frac{1}{\tau^2} \left(\frac{m^2}{H^2} - 2 \right)$$

$$\delta\sigma_k + (k + M(z))\delta\sigma_k = 0$$

can be recast in the form

$$\delta\sigma_k'' + \left[k^2 - \frac{1}{z^2} \left(\nu_k^2 - \frac{1}{4} \right) \right] \delta\sigma_k = 0$$

where
$$\nu_k^2 = \frac{1}{4} - \frac{m_k^2}{H^2}$$

The general solution for ν_k real is

$$\delta\sigma_k = \sqrt{-z} \left[c_1(k) H_{\nu_k}^{(1)}(-kz) + c_2(k) H_{\nu_k}^{(2)}(-kz) \right]$$

where $H_{\nu_k}^{(1)}$ and $H_{\nu_k}^{(2)}$ are the Hankel's functions of the first and second kind.

Impose that in the ultraviolet regime $k \gg aH$ ($-kz \gg 1$) the solution matches the plane wave solution $e^{ikz}/\sqrt{2k}$ that we expect in flat spacetime

$$H_{\nu_k}^{(1)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}} e^{i(x - \frac{\pi}{2}\nu_k - \frac{\pi}{4})}$$

$$H_{\nu_k}^{(2)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}} e^{-i(x - \frac{\pi}{2}\nu_k - \frac{\pi}{4})}$$

We set $c_2(k) = 0$, $c_1(k) = \frac{\sqrt{\pi}}{2} e^{i(\nu_k + \frac{1}{2})\pi/2}$, the solution

becomes
$$\delta\sigma_k = \frac{\sqrt{\pi}}{2} e^{i(\nu_k + \frac{1}{2})\pi/2} \sqrt{-z} H_{\nu_k}^{(1)}(-kz)$$

On superhorizon scales, since

$$H_{U_k}''' (\chi \ll 1) \sim \sqrt{2\pi} e^{-i\frac{\pi}{2}} 2^{U_k - \frac{3}{2}} \left(\Gamma(U_k) / \Gamma(3/2) \right) \chi^{-U_k}.$$

the solution becomes

$$\delta\sigma_k = e^{i(U_k - \frac{1}{2})\frac{\pi}{2}} 2^{(U_k - \frac{3}{2})} \frac{\Gamma(U_k)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\tau)^{\frac{1}{2} - U_k}$$

Going back to the old variable $\delta\chi_k$ ($\delta\chi_k = \frac{\delta\sigma_k}{a}$) we find that on superhorizon scales, the fluctuation with nonvanishing mass is not exactly constant, but it acquires a tiny dependence on the time

$$|\delta\chi_k| \approx \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH} \right)^{\frac{3}{2} - U_k}$$

c.f. $|\delta\chi_k| \approx \frac{H}{\sqrt{2k^3}}$ (massless)

3) Quantum fluctuations of a generic scalar field in a quasi de Sitter stage

We have computed the time evolution of the quantum fluctuations of a generic scalar field ϕ supposing that the scale factor evolves like in a

pure de Sitter expansion, $a(t) = -1/(HE)$.

During inflation the Hubble rate is not exactly constant, but changes with time as $\dot{H} = -\epsilon H^2$ (quasi de Sitter expansion)

Perturbations in a quasi de Sitter expansion using the definition of the conformal time, the scale factor for small values of ϵ becomes

$$a(\tau) = -\frac{1}{H} \frac{1}{\tau(1-\epsilon)}$$

$$\delta\sigma_k'' + [k^2 + M^2(\tau)]\delta\sigma_k = 0$$

with $M^2(\tau) = m_\chi^2 a^2 - \frac{\epsilon''}{a^2}$ now

where

$$\begin{aligned} \frac{\epsilon''}{a^2} &= a^2 \left(\frac{\ddot{\epsilon}}{a} + H^2 \right) = a^2 (H + 2H^2) \\ &= a^2 (2-\epsilon) H^2 = \frac{(2-\epsilon)}{\tau^2 (1-\epsilon)^2} \approx \frac{1}{\tau^2} (2+3\epsilon) \end{aligned}$$

Taking $m_\chi^2/H^2 = 3\eta_\chi$ and expanding for ϵ and η .

we get
$$\delta\sigma_k'' + [k^2 - \frac{1}{\tau^2} (U_\chi^2 - \frac{1}{4})] \delta\sigma_k = 0$$

with
$$U_\chi \approx \frac{3}{2} + \epsilon - \eta_\chi$$

Armed with these results, we may compute the variance of the perturbations of the generic χ field

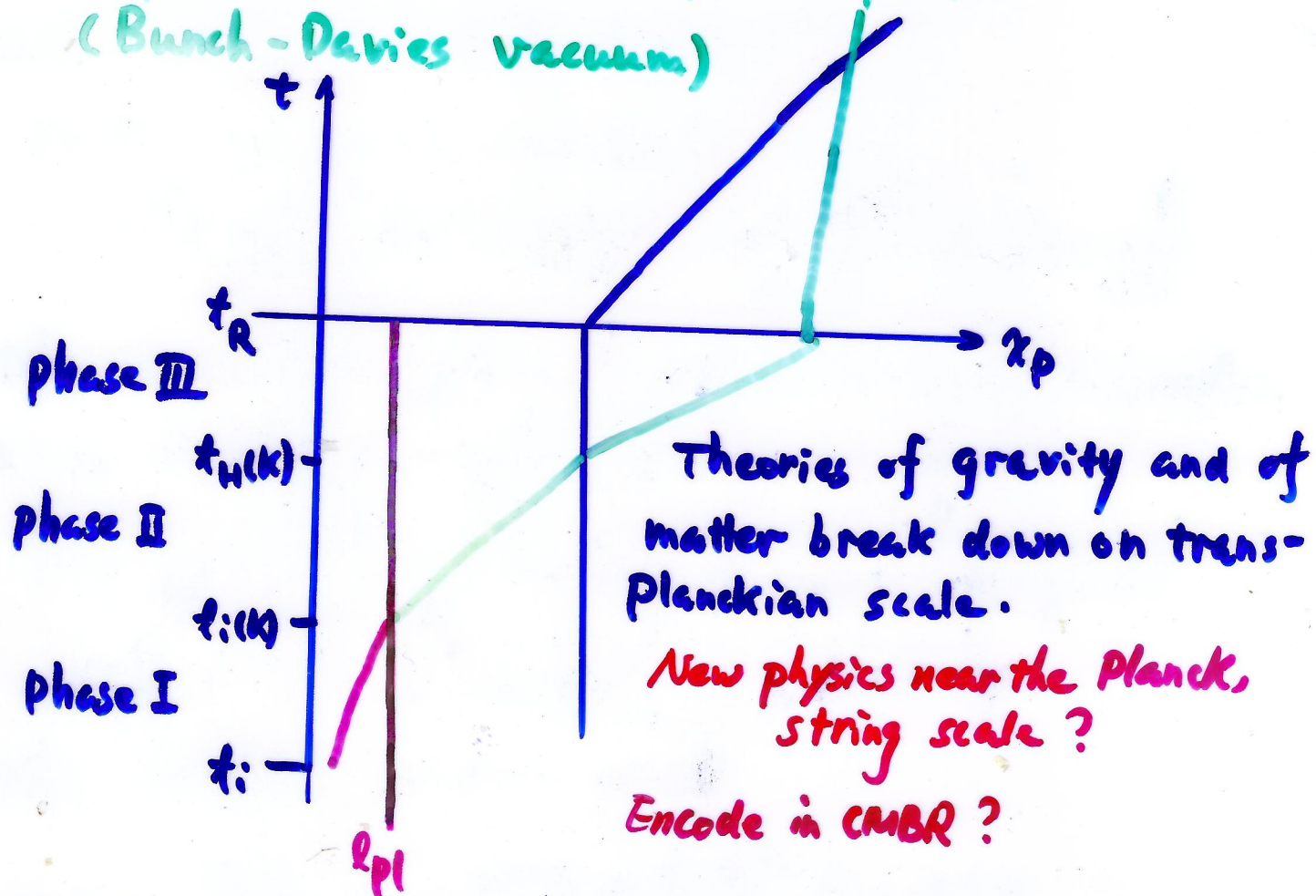
3. The Trans-Planckian Window

Standard Inflationary scenario:

Initial conditions for the inflaton field are imposed in the infinite past when the modes are infinitely small.

In this limit the effect of the inflationary horizon and the expansion of the universe can be ignored, the spacetime is essentially Minkowski, and there is a unique vacuum for the inflaton field.

(Bunch-Davies vacuum)



Phase I: wavelength is smaller than the Planck scale l_{Pl}

The new physics can be encoded in the choice of initial conditions when the mode of the inflaton has a wavelength comparable to the scale where new physics is expected. (Danielsson)

Bunch - Davies vacuum Not necessarily UNIQUE

1. Example :

In the Standard FRW coordinates

$$ds^2 = dt^2 - c^2(t) dx^2$$

$a(t) = e^{Ht}$ is the scale factor

In the conformal coordinate

$$ds^2 = \frac{1}{H^2 \eta^2} (d\eta^2 - dx^2) \quad d\eta = \frac{dt}{a} \Rightarrow \eta = -\frac{1}{cH}$$

The physical momentum P and the comoving momentum k are related through $k = ap = -\frac{P}{\eta H}$

The equation of motion of the scalar field

$$\mu_k'' + (k^2 - \frac{a''}{a}) \mu_k = 0 \quad (\mu = a\phi)$$

The general solution reads

$$\mu = D_1 \eta^{1/2} J_{-3/2}(k\eta) + D_2 \eta^{1/2} J_{3/2}(k\eta)$$

$$\underline{k\eta \gg 1} \quad C_+ e^{i(k\eta)} + C_- e^{-i(k\eta)}$$

Boundary condition

1) Relation between the field and its conjugate momentum

Bunch-Davies vacuum:

$$\frac{d\phi_p}{dt} \rightarrow -iP\phi_p. P \rightarrow \omega \Rightarrow \frac{k}{a} \rightarrow \omega \Rightarrow \lambda_{\text{pl}} \rightarrow 0$$

Now: $\frac{d\phi_p}{dt} \rightarrow -iP\phi_p. P = P_c = \Lambda$ Λ : energy scale for New Physics

2) Wronskian condition $|C_+|^2 - |C_-|^2 = 1$

we have

$$C_+ = \frac{2k\eta + i}{2k\eta} e^{-ik\eta}$$

$$C_- = \frac{-i}{2k\eta} e^{ik\eta}$$

$$k\eta \rightarrow \infty \Rightarrow \begin{matrix} C_+ = 1 \\ C_- = 0 \end{matrix} \text{ B-D vacuum}$$

$$P = |C_+ + C_-|^2 P^{\text{B-D}} = \left(\frac{H}{2\pi}\right)^2 \left[1 - \frac{H}{\Lambda} \sin\left(\frac{2\Lambda}{H}\right) \right]$$