

## Klein's Paradox and Its Resolution

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# Klein's Paradox and Its Resolution

Alex Hansen and Finn Ravndal

Institute of Physics, University of Oslo, Oslo 3, Norway

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## Abstract

Klein's paradox is resolved in simple diagrammatic terms arising from considerations of its analogue in classical relativistic mechanics. The same results are also obtained directly from standard operator methods of quantum field theory.

## 1. Introduction

In the last few years the physics behind Klein's paradox [1] has gained new relevance in connection with the production of elementary particles in strong electromagnetic and gravitational fields [2]. This new interest in the old paradox has been especially spurred by the surprising and beautiful discovery of Hawking [3] which leads to the evaporation of black holes by particle emission.

As most paradoxes in physics, Klein's paradox was resolved a long time ago dating back to the investigations of Sauter [4], Sommerfeld [5] and Hund [6]. The resulting explanations are based on the effects of spontaneous production of particle-antiparticle pairs when the electric potential differences get large enough. But since pair production cannot be described within one-particle or first quantized Dirac theory a completely satisfactory resolution of the paradox must be given within second quantized field theory.

In this more modern language the problems arising in external electromagnetic fields have been discussed by many and in particular by Feynman [7, 8]. His point of view has more recently been carried further in great detail by Nikishov [9, 10] and Narozhnyi and Nikishov [11]. These calculations give a clear explanation of the physics involved in Klein's process and there is no room left for any paradox.

One reason why the paradox still lingers on is that its detailed resolution is not well known and does not yet appear in any textbook on quantum field theory. For instance, in the highly recommended book by Bjorken and Drell [12] the paradox is presented in the same way as Klein originally did it within the one-particle Dirac theory. A resolution of the paradox is promised later in the text, but no explicit discussion of this particular problem can be found.

A second reason why this phenomenon is still considered paradoxical is that its modern resolution has apparently little connection with the original formulation of the problem. This is obviously because the paradox is stated within ordinary relativistic quantum mechanics and must be resolved by the methods of second quantized field theory.

We will here try to bridge the gap between these two sides of the problem. One step in this direction has already been taken by Damour [13] whose point of view is similar to ours. In the next section we state the problem as originally done by Klein. For completeness, we also consider the related problem involving bosons. Some early attempts at the resolution of the para-

dox are also mentioned. By going back to the non-quantum version of the same physical problem we show in Section 3 that when the potential becomes strong enough, a relativistic particle is forced to move backwards in time. This purely classical result must be automatically incorporated in the quantum mechanical description. In Section 4 this is done in terms of simple diagrams very much along the same lines as in ordinary geometrical optics. The possibility of spontaneous pair production of particles arises then quite naturally. These diagrammatic results are reproduced in Section 5 where we use standard methods of quantum field theory to derive the Bogoliubov relations between ingoing and outgoing operators. In Section 6 we use these operators to calculate the S-matrix elements for scattering and pair production in external fields. These results are in full agreement with what we obtained more directly from the diagrams and there is no longer any paradox.

## 2. Klein's paradox

Consider a free particle moving along the  $z$ -axis with energy  $E$  and electric charge  $e$  hitting a static potential step of height  $V$  as shown in Fig. 1. Its momentum to the left of the step is

$$p = \sqrt{E^2 - m^2} \quad (1)$$

and to the right it is

$$q = \sqrt{(E - eV)^2 - m^2} \quad (2)$$

where we take the positive square roots. In order to calculate the reflected and transmitted waves we match the wavefunctions and their first derivatives across the step at  $z = 0$  when the incident particles are bosons. For incident fermions we similarly match the upper and lower components of the one-particle spinors. In both cases we find a reflection coefficient

$$\rho(q) = \left( \frac{1 - \kappa}{1 + \kappa} \right)^2 \quad (3)$$

and a transmission coefficient

$$\tau(q) = \frac{4\kappa}{(1 + \kappa)^2} \quad (4)$$

where

$$\kappa = \frac{q}{p} \quad (5)$$

for bosons and

$$\kappa = \frac{q}{p} \frac{E + m}{E - eV + m} \quad (6)$$

for fermions.

For weak potentials where  $E - eV + m > 0$  the parameter  $\kappa > 0$  and we have the relation

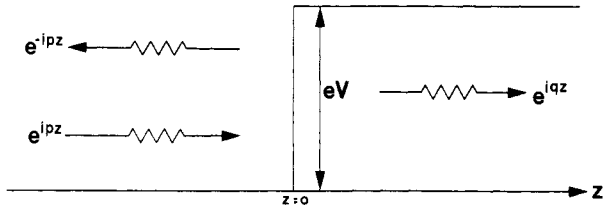


Fig. 1. Potential step with incoming particles from the left.

$$\rho + \tau = 1 \quad (7)$$

both for fermions and bosons. This is just an expression for probability conservation and no problems arise.

More interesting are very strong potentials where  $E - eV + m < 0$ , i.e.,  $eV > 2m$ . This is the situation first considered by Klein and which we will be concerned with in the following. We see now from eq. (6) that  $\kappa < 0$  for fermions. This implies a negative transmission coefficient and a reflection coefficient  $\rho > 1$ . There will apparently be more fermions reflected than coming in. This is Klein's paradox. Its resolution must be found in an understanding of the negative transmission coefficients which corresponds to particles coming from the right in Fig. 1. For bosons we don't have this paradoxical situation.

An indication of what goes wrong can be seen from the group velocity of the particles within the potential. From the momentum (2) we find

$$v = \frac{dE}{dq} = \frac{q}{E - eV} \quad (8)$$

In the strong field case this velocity is therefore negative. This is physically meaningless since particles in this region must move to the right. We can get a positive velocity by choosing the negative square root in eq. (2), i.e., by letting  $q < 0$ . At the same time the parameter  $\kappa$  for fermions becomes positive. Thus it looks like the paradox is resolved. Unfortunately, the problem has just been transferred to the bosons which for the same kind of reasons should have  $q < 0$ . This makes  $\kappa < 0$  and hence  $\rho > 1$  for bosons.

It is natural to look for an explanation of these problems in the possibility of production of particle-antiparticle pairs which kinematically can take place when  $eV > 2m$ . This was also the point of view taken by Sommerfeld [5] although he tried to work within the first quantized formalism. A more satisfactory description was given by Hund [6] who apparently for the first time took into account the simplest consequences of a second quantized formalism. He could then explicitly show that when  $eV > 2m$  particle pairs would be produced with a rate simply given by the transmission coefficient for negative momentum:

$$\bar{n} = |\tau(-q)|^2 \quad (9)$$

It is not at all obvious how such a result can be obtained within conventional quantum electrodynamics whose calculational framework was completed several years later. In addition, one does not get a very clear understanding of what is going on in the strong field case from Hund's considerations. We will therefore in the next section consider the problem at the classical level in order to get a physical picture for what is happening.

### 3. Classical particles in strong fields

The problem under consideration was discussed classically a long time ago by Feynman [14]. It is illustrated again in a Min-

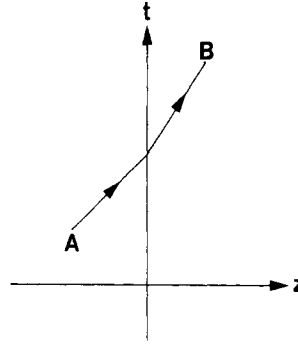


Fig. 2. A relativistic particle going from A to B is bent into the potential when it is not too strong.

kowski diagram in Fig. 2. A relativistic particle is coming in from the left along the  $z$ -axis moving towards a potential step of height  $V$ . The covariant Lagrangian is

$$L = \frac{m}{2} \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - e A_\mu \dot{x}^\mu \quad (10)$$

where  $\eta_{\mu\nu}$  is the Minkowski metric,  $A_\mu$  the electromagnetic 4-potential and the dot derivatives are with respect to proper time  $s$ . In our case this Lagrangian reduces to

$$L = \frac{m}{2} (\dot{z}^2 - \dot{t}^2) - eV\dot{t} \quad (11)$$

The equations of motions now tell us that the particle moves with constant velocity in the regions where  $V = \text{constant}$ . In order to find the actual trajectory of the particle when it moves from point A to point B in Fig. 2 we must extremize the action

$$S = \int_A^B ds L \quad (12)$$

with respect to the time  $t$  at which the particle crosses the barrier at  $z = 0$ . It is much simpler to use conservation of energy

$$E = -\frac{\partial L}{\partial \dot{t}} \quad (13)$$

which gives the same result

$$\dot{t}_A = \dot{t}_B + \frac{eV}{m} \quad (14)$$

Free particles have energies  $E > m$ , i.e.,  $\dot{t} \geq 1$ . In the weak potential case,  $eV < m$ , eq. (14) gives the kinetic energy  $m\dot{t}_B$  of the particle within the potential. Obviously,  $\dot{t}_B < \dot{t}_A$  as is the case in Fig. 2. When the potential difference gets larger,  $m < eV < 2m$ , then  $\dot{t}_B < 1$  which corresponds to imaginary velocity for the particle to the right of the step. Classically, this is not an acceptable solution and we conclude that the particle now will not enter the potential but is simply reflected back.

When  $eV > 2m$ , we again find real velocities within the potential. But now  $\dot{t}_B < -1$  which means that the particle is moving backwards in time as shown in Fig. 3. In this figure we have also included the trajectory of a reflected particle which now also gives a stationary action.

This surprising classical result is obviously a relativistic effect. The physical interpretation of such a particle trajectory moving backwards in time, was first discussed by Stückelberg [15] and Feynman [7, 14]. They showed that one obtains a consistent picture if these trajectories correspond to antiparticles moving forwards in time. Accepting this point of view, we have to con-

clude that the incoming particle seems to set up a corresponding antiparticle within the potential which it later can annihilate. A quantitative description of this process which apparently also breaks causality must be found within quantum theory.

4. Elementary quantum processes in strong fields

In following it will be convenient to introduce a reflection amplitude  $R = \sqrt{\rho}$  and a transmission amplitude  $T = \sqrt{\tau}$ . We will keep  $q > 0$  in the potential region  $z > 0$ .

When  $eV > 2m$  current conservation as expressed by eq. (7) then gives for bosons

$$|R|^2 + |T|^2 = 1 \tag{15}$$

Similarly, for fermions

$$|R|^2 - |T|^2 = 1 \tag{16}$$

where  $T$  now is imaginary.

Solutions of the wave equations which in the region  $z < 0$  behave as  $p_1 \sim e^{ipz}$  and  $p_2 \sim e^{-ipz}$  correspond to incoming and outgoing particles. Similarly, solutions which in the right region  $z > 0$  behave as  $n_1 \sim e^{ipz}$  and  $n_2 \sim e^{-iqz}$  correspond to incoming and outgoing antiparticles. These solutions are given in more detail in the Appendix.

The process under consideration in Fig. 3 involves an incoming particle wave  $p_1$  coming in from the left and which is partly reflected back in space into  $p_2$  and partly refracted back in time into  $n_1$ . This latter wave with negative frequency corresponds to an antiparticle coming in from the right. The different wave functions are related by

$$p_1 + Rp_2 = Tn_1 \tag{17}$$

since there are no outgoing antiparticles within the potential.

There are two physical conclusions we can make from this equation. First of all there is a scattering probability

$$\sigma = |R(q)|^2 \tag{18}$$

corresponding to the Feynman diagram shown in Fig. 4(a). Secondly, there is a probability

$$\omega = |T(q)|^2 \tag{19}$$

for the incoming particle to annihilate the incoming antiparticle as in Fig. 4(b). This must also be the probability for spontaneous production of a single particle-antiparticle pair at the potential step as shown in the Feynman diagram Fig. 4(c). Energetically, there is nothing to prevent many such pairs to be created as long as the strength of the potential is maintained.

In order to calculate the amount of produced antiparticles,

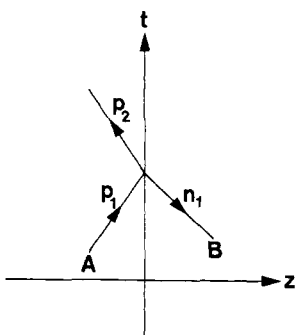


Fig. 3. When the potential is stronger than  $eV > 2m$ , the relativistic particle is scattered forward in time or refracted backward into the past.

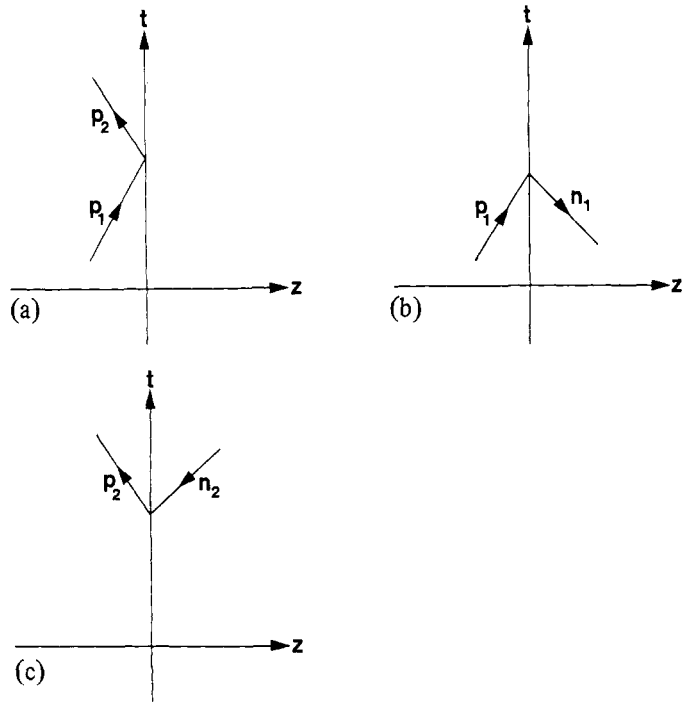


Fig. 4. Elementary processes in a strong potential. (a) Elastic scattering. (b) Pair annihilation. (c) Pair production.

we must specify the conditions under which they are produced. If we insist that there should only be an incoming particle upon the potential and no antiparticles from the right, we must extinguish the  $n_1$  wave in the  $z > 0$  region of Fig. 3. This can easily be done by adding in another solution of the wave equation corresponding to negative frequency wave  $n_2$  incoming from the future. As shown in Fig. 5 it is partly reflected back into the past and partly refracted forward in time to become an outgoing positive frequency wave in the left region. Matching again wave functions at the boundary  $z = 0$ , we now find for bosons

$$n_2 - R^*n_1 = T^*p_2 \tag{20}$$

The extra minus sign in the reflected wave results from the interchange of momenta  $p$  and  $q$  in eq. (3). We have in this equation included the more general case when for instance the potential step is not at the origin of the coordinate system in Fig. 1 or when the potential has a different shape which makes both  $R$  and  $T$  complex.

We can eliminate the incoming antiparticle in Fig. 3 by subtracting this new solution (20) multiplied by  $T$  from the previous solution (17) multiplied by  $R^*$ . Using eq. (15) we then find:

$$p_1 + \frac{1}{R^*}p_2 = \frac{T}{R^*}n_2 \tag{21}$$

For fermions we obtain the same result since eq. (20) then will have an extra minus sign on the right-hand side due to  $T$  being imaginary for fermions.

From eq. (21) we now find the average rate of the outgoing antiparticles:

$$\bar{n} = \frac{|T(q)|^2}{|R(q)|^2} = |T(-q)|^2 \tag{22}$$

This is in agreement with Hund's result (9). A more convincing derivation will now be given.

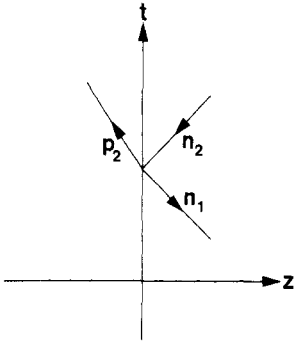


Fig. 5. Negative frequency solution coming in from the future and being scattered backward in time or refracted forward into the future.

Each pair is produced with the probability  $\omega$  from eq. (19) so that the average number of bosons pairs becomes:

$$\bar{n} = C[\omega + 2\omega^2 + 3\omega^3 + \dots] = \frac{C\omega}{(1-\omega)^2} \quad (23)$$

Here  $C$  is the probability that the vacuum remains a vacuum. It can be calculated from the requirement that the sum of the probabilities for all that can happen, must be one:

$$1 = C[1 + \omega + \omega^2 + \dots] = \frac{C}{1-\omega} \quad (24)$$

Using  $C = 1 - \omega$  in eq. (23) we find again

$$\begin{aligned} \bar{n} &= \frac{\omega}{1-\omega} \\ &= \frac{|T(q)|^2}{|R(q)|^2} = |T(-q)|^2 \end{aligned} \quad (25)$$

as in eq. (22).

For fermions we obtain the same result. The Pauli principle then allows only one pair to be produced in a given state so that

$$\bar{n} = C\omega \quad (27)$$

Instead of eq. (24) we now have  $1 = C(1 + \omega)$  which gives

$$\bar{n} = \frac{\omega}{1+\omega} \quad (28)$$

With  $\omega$  from eq. (19) together with eq. (16), we again find eq. (26).

It should be stressed that all this holds for each particular state the particles can be found in. In the one-dimensional case under consideration each such mode is completely specified by the total particle energy  $E$ .

## 5. Bogoliubov coefficients

Given a static potential  $V$  which only varies in the  $z$ -direction, we assume that we can find the one-particle states which are solutions of the corresponding Klein-Gordon or Dirac equation. We call the positive frequency solutions  $p_k(x)$  and the negative frequency solutions  $n_k(x)$ . As for the step potential, these are specified by their behaviour at  $x \rightarrow \pm \infty$ . The index  $k$  denotes the conserved quantum numbers of the corresponding modes which here in general will be the total particle energy  $E$  and its transverse momentum  $p_T$ .

Both for fermions and bosons one can define a scalar product of two solutions  $u$  and  $v$  such that

$$(u, v) = (v, u)^* \quad (29)$$

It is then convenient to use the standard orthonormalized solutions satisfying

$$(p_k, p_l) = \delta_{kl}, \quad (p_k, n_l) = 0, \quad (n_k, n_l) = \epsilon \delta_{kl} \quad (30)$$

where  $\epsilon$  is  $+1$  for fermions and  $-1$  for bosons.

As in all scattering problems we will actually find two such complete sets of solutions depending on their behaviour at infinity. These two sets will be labelled by an extra index taking the value 1 for incoming waves and 2 for outgoing waves. A field operator  $\psi(x)$  can then be expanded in these two sets to give

$$\psi(x) = \sum_k [a_{1k} p_{1k}(x) + b_{1k}^* n_{1k}(x)] \quad (31a)$$

or

$$\psi(x) = \sum_k [a_{2k} p_{2k}(x) + b_{2k}^* n_{2k}(x)] \quad (31b)$$

Here the operator  $a_{1k}$  will annihilate an incoming particle with the wavenfunction  $p_{1k}(x)$  while  $b_{2k}^*$  will create an outgoing anti-particle with the wavefunction  $n_{2k}(x)$ .

All the physical transitions which we will consider will be between states with the same conserved quantum numbers. The corresponding equations will therefore hold for each separate mode label  $k$ . For convenience, we will henceforth drop it.

Using the orthonormalization (30) in eq. (31) we can now relate the two sets of incoming and outgoing field operators:

$$a_1 = Aa_2 + Bb_2^* \quad (32a)$$

$$\epsilon b_1^* = \tilde{B}a_2 + \tilde{A}b_2^* \quad (32b)$$

$$a_2 = A^*a_1 + \tilde{B}^*b_1^* \quad (32c)$$

$$\epsilon b_2^* = B^*a_1 + \tilde{A}^*b_1^* \quad (32d)$$

Here we have introduced the Bogoliubov coefficients

$$A = (p_1, p_2), \quad B = (p_1, n_2) \quad (33)$$

$$\tilde{A} = (n_1, n_2), \quad \tilde{B} = (n_1, p_2) \quad (34)$$

and their complex conjugates.

The boson theory is quantized by commutators and the fermion theory by anticommutators. For example,

$$[a_1, a_1^\dagger]_\epsilon = 1, \quad [b_1, b_1^\dagger]_\epsilon = 1 \quad (35)$$

where again  $\epsilon = +1$  for fermions and  $\epsilon = -1$  for bosons. Equation (32a) then gives immediately

$$|A|^2 + \epsilon|B|^2 = 1 \quad (36)$$

The same relation obviously also holds for the Bogoliubov coefficients  $\tilde{A}$  and  $\tilde{B}$ . These differ from  $A$  and  $B$  just by phases. Similarly, from  $[a_2, b_2]_\epsilon = 0$  follows

$$AB^* = -\epsilon \tilde{A}^* \tilde{B} \quad (37)$$

Using the operator relations (32) in the field operator (31) we can solve for the wave functions:

$$p_1 = A^* p_2 + \epsilon B^* n_2 \quad (38a)$$

$$n_1 = \tilde{B}^* p_2 + \epsilon \tilde{A}^* n_2 \quad (38b)$$

$$p_2 = A p_1 + \epsilon \tilde{B} n_1 \quad (38c)$$

$$n_2 = B p_1 + \epsilon \tilde{A} n_1 \quad (38d)$$

These Bogoliubov relations are of exactly the same form as we previously derived in eqs. (17), (20) and (21) for the potential step. Comparing for instance, eq. (38c) with eq. (17), we find

$$R = -\frac{1}{A} \tag{39}$$

and

$$T = -\epsilon \frac{\tilde{B}}{A} \tag{40}$$

For similar potentials with the same left-right symmetry we can show that there are only two independent Bogoliubov coefficients:

$$\tilde{A} = -\epsilon A, \quad \tilde{B} = -\epsilon B \tag{41}$$

The transmission coefficient can therefore also be written as

$$T = \frac{B}{A} \tag{42}$$

In terms of these previously used coefficients the commutator relation (36) now takes the form

$$|R|^2 - \epsilon |T|^2 = 1 \tag{43}$$

as expected. Finally, the constraint (37) can be written as

$$T = -\epsilon T^* \tag{44}$$

For fermions with  $\epsilon = +1$  we get a purely imaginary transmission coefficient. This is also in accordance with the phase convention we used previously.

Another way to understand Klein's apparent paradox is suggested by eq. (43). For fermions, it implies that  $|R|^2 \geq 1$ . We now see this fundamental result as a direct consequence of the fact that fermions obey the Pauli principle and hence are quantized by anticommutators.

### 6. S-matrix elements

The incoming and outgoing operators can be used to construct two complete sets of incoming and outgoing states. These are as usually related by the unitary S-matrix operator S:

$$|\Psi_1\rangle = S|\Psi_2\rangle \tag{45}$$

The incoming state  $|\Psi_1\rangle$  is obtained by applying incoming creation operators on the incoming vacuum  $|O_1\rangle$ . For example, an incoming one-particle state with quantum number  $p$  is

$$|p_1\rangle = a_{1p}^+|O_1\rangle = Sa_{2p}^+|O_2\rangle \tag{46}$$

using

$$|O_1\rangle = S|O_2\rangle \tag{47}$$

we derive the corresponding relation between the two sets of operators:

$$a_1 = Sa_2S^+ \tag{48}$$

The two vacua are both annihilated by their corresponding destruction operators:

$$a_1|O_1\rangle = a_2|O_2\rangle = 0 \tag{49}$$

Together with eq. (32a) we now find the relation

$$a_2|O_1\rangle = -\frac{B}{A}b_2^+|O_1\rangle \tag{50}$$

which can be used to find the amplitude for spontaneous pair production in the potential corresponding to the process in Fig. 4(c):

$$S_{\text{pair}} = \langle O_2|b_2a_2|O_1\rangle = e^{iW} \left( -\frac{B}{A} \right) \tag{51}$$

Here we have introduced the probability amplitude

$$e^{iW} = \langle O_2|O_1\rangle \tag{52}$$

for the vacuum to remain a vacuum under the influence of the external potential. The absolute probability for pair production is therefore

$$S_{\text{pair}}^2 = C\omega \tag{53}$$

where  $\omega$  is the relative probability (19) and

$$C = e^{-2\text{Im} W} \tag{54}$$

This is obviously in full agreement with what we found in Section 4.

In the same way we can calculate the absolute probability amplitude for elastic scattering as shown in Fig. 4(a):

$$S_{\text{scatt}} = \langle O_1|a_2a_1^+|O_1\rangle \tag{55}$$

Inserting here  $a_1^+$  from eq. (32a), we get

$$S_{\text{scatt}} = A^*\langle O_2|O_1\rangle + B^*\langle O_2|a_2b_2|O_1\rangle \tag{56}$$

The first term is what we could call the direct scattering term since it corresponds to the  $1/R^*$  term in eq. (21). But it is seen to be corrected by the pair production amplitude (51) which corresponds to the  $T/R^*$  term in eq. (21). Again it comes in because we do not want any incoming antiparticles in the initial state.

We combine the two terms in eq. (56) and find

$$S_{\text{scatt}} = e^{iW} \frac{1}{A} \tag{57}$$

both for fermions and bosons. The absolute, elastic scattering probability is therefore

$$|S_{\text{scatt}}|^2 = C\sigma \tag{58}$$

where  $\sigma$  is the relative scattering probability (18).

Again we want to emphasize that the last pair production term in the scattering amplitude (56) is in close correspondence with the cancellation of the incoming antiparticle solution we had to perform in going from eq. (19) to eq. (21) in Section 4. We have now a quantum field operator justification for the diagrammatic considerations we presented there.

Similarly, we derive the probability conservation in eqs. (24) and (29) directly from the unitarity of the S-matrix,

$$S^*S = 1 \tag{59}$$

by inserting a complete set of outgoing states and then take both sides of this equation between  $\langle O_2|$  and  $|O_2\rangle$ . For the potential problem we consider here, unitarity holds separately for each particular mode specified by energy and transverse momentum. As previously, we have here suppressed the indices denoting these conserved quantum numbers.

The average rate of spontaneously produced pairs of particles in a certain mode is now simply given as

$$\bar{n} = \langle O_1|a_2^+a_2|O_1\rangle \tag{60}$$

With  $a_2$  from eq. (32c) we find

$$\bar{n} = |B|^2 = \frac{|T|^2}{|R|^2} \tag{61}$$

when using eqs. (39) and (40). We have again derived Hund's result (26) which is seen to hold for both fermions and bosons.

## 7. Discussion and conclusion

It can now be said that Klein's paradox arises from the Dirac equation basically because of the Pauli principle. Even the one-particle solutions carry the properties that they can only consistently be used as probability amplitudes in a field theory which is quantized by anticommutators.

The paradox is also resolved by strictly satisfying the Pauli principle. This has the consequence that when we scatter a fermion with a certain energy from a static potential  $eV > 2m$ , it will simply be totally reflected. This particular outgoing mode will then be occupied and no pair production of particles with the same energy can take place. Since this is the only possible process, it must have absolute probability (58) equal to one:

$$C\sigma = 1 \quad (62)$$

From eq. (29) we find  $C^{-1} = |R|^2$  and since  $\sigma = |R|^2$  we see that the equation is indeed satisfied.

An incoming boson on the same potential will also be totally reflected. But this boson will now stimulate the emission of pairs in the same energy mode. So in addition to the reflected boson, there can appear any number of produced pairs with the same quantum numbers. The probability amplitude to have  $n$  additional pairs can be written down immediately as in eq. (55):

$$S_{\text{scatt}}^{(n)} = (n!(n+1)!)^{-1/2} \langle O_2 | b_2^n a_2^{n+1} a_1^+ | O_1 \rangle$$

where we have included the normalization factor of the final state. Again using  $a_1$  from (32a), this reduces to

$$S_{\text{scatt}}^{(n)} = \sqrt{n+1} e^{iW} \left[ A^* + B^* \left( -\frac{B}{A} \right) \right] \left( -\frac{B}{A} \right)^n$$

The corresponding probability is therefore

$$|S_{\text{scatt}}^{(n)}|^2 = C\sigma(n+1)\omega^n \quad (63)$$

But the sum of the probabilities for all these different final states is again equal to one:

$$C\sigma \sum_{n=0}^{\infty} (n+1)\omega^n = \frac{C\sigma}{(1-\omega)^2} = 1 \quad (64)$$

since the vacuum-to-vacuum probability  $C$  is now given by eq. (25). Probability conservation is therefore also satisfied for bosons.

Finally, we want to conclude with a short discussion of Sauter's potential [4]:

$$V(z) = \frac{V}{2} \tanh \frac{az}{2} \quad (65)$$

This is a smooth version of Klein's step potential and allows exact solutions. For fermions these were obtained by Sauter himself. He found the following reflection and transmission probabilities:

$$\rho(q) = \frac{\cosh(2\pi/a)eV - \cosh(2\pi/a)(p-q)}{\cosh(2\pi/a)eV - \cosh(2\pi/a)(p+q)} \quad (66)$$

$$\tau(q) = \frac{\cosh(2\pi/a)(p-q) - \cosh(2\pi/a)(p+q)}{\cosh(2\pi/a)eV - \cosh(2\pi/a)(p+q)} \quad (67)$$

We see that  $\tau(q) < 0$  which is the signal of Klein's paradox.

In the limit  $a \rightarrow \infty$  we recover the original results (3) and (4) for the step potential. The limit  $a \rightarrow 0$ ,  $V \rightarrow \infty$  with  $aV \rightarrow 4E$  is more interesting. It corresponds to a linear potential due to a constant electric field  $E$ . The probability for spontaneous pair creation can now be obtained from eq. (67):

$$\omega = |T|^2 = -\tau(q) = \frac{\exp(-\pi m^2/eE)}{1 - \exp(-\pi m^2/eE)} \quad (68)$$

Comparing with eq. (30) we find for the average number of produced pairs in each mode:

$$\bar{n} = \exp(-\pi m^2/eE) \quad (69)$$

This is the same as found later by Schwinger [16] using completely different methods. The same result for  $\bar{n}$  can also be obtained for bosons in the constant field limit. Inclusion of transverse momentum  $p_T$  amounts to the replacement  $m^2 \rightarrow m^2 + p_T^2$  in the above expressions.

The resolution of Klein's paradox is due to the spontaneous production of particle-antiparticle pairs in the static potential. One can argue that the problem is not really static and that the pairs are produced when the potential is turned on. This implicit time variation is buried behind the adiabatic hypothesis which must be invoked in order to make the  $S$ -matrix formulation well-defined. What actually happens when the potential is turned on, can be calculated in the case of the Sauter potential (65). We expect to find our static results long after the switch-on. It should then not matter how the potential is turned on. These calculations are now in progress.

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## Appendix

We will here present the solutions we have used for the step potential. For both bosons and fermions they can be written as:

$$p_1(z) = \begin{cases} P_1, & z < 0 \\ \frac{1+\kappa}{2\sqrt{\kappa}} N_1 - \frac{1-\kappa}{2\sqrt{\kappa}} N_2, & z > 0 \end{cases}$$

$$p_2(z) = \begin{cases} P_2, & z < 0 \\ -\frac{1-\kappa}{2\sqrt{\kappa}} N_1 + \frac{1+\kappa}{2\sqrt{\kappa}} N_2, & z > 0 \end{cases}$$

$$n_1(z) = \begin{cases} \frac{1+\kappa}{2\sqrt{\kappa}} P_1 + \frac{1-\kappa}{2\sqrt{\kappa}} P_2, & z < 0 \\ N_1, & z > 0 \end{cases}$$

$$n_2(z) = \begin{cases} \frac{1-\kappa}{2\sqrt{\kappa}} P_1 + \frac{1+\kappa}{2\sqrt{\kappa}} P_2, & z < 0 \\ N_2, & z > 0 \end{cases}$$

For bosons,

$$P_1 = \frac{1}{\sqrt{|p|}} e^{ipz}$$

$$N_1 = \frac{1}{\sqrt{|q|}} e^{iqz}$$

while for fermions:

$$P_1 = \sqrt{\frac{E+m}{|p|}} \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E+m} \\ 0 \end{pmatrix} e^{ipz}$$

$$N_1 = \sqrt{\frac{E-eV+m}{|q|}} \begin{pmatrix} 1 \\ 0 \\ \frac{q}{E-eV+m} \\ 0 \end{pmatrix} e^{iqz}$$

As in the main text,  $p$  and  $q$  are both positive while  $\kappa$  is positive for bosons and negative for fermions. Outgoing waves  $P_2$  and  $N_2$  are obtained from these expressions by letting  $p \rightarrow -p$  and  $q \rightarrow -q$ .

The inner product of these solutions are for bosons given by

$$(u, v) = i \int_{-\infty}^{+\infty} dz u^* \vec{\nabla}_0 v$$

where

$$\vec{\nabla}_0 = \vec{\partial}_0 - 2ieV(z)$$

In the case of massless particles these integrals are pretty straightforward to do. We find:

$$(p_1, p'_1) = -\pi \frac{(p-q)^2}{pq} \delta(p-p')$$

$$(p_1, n'_1) = 0$$

$$(p_1, p'_2) = +\pi \frac{p^2 - q^2}{pq} \delta(p-p')$$

$$(p_1, n'_2) = 2\pi \frac{p-q}{\sqrt{pq}} \delta(p-p')$$

The resulting Bogoliubov coefficients are in agreement with those previously obtained in the text from the reflection and transmission amplitudes.

For fermions, one can similarly obtain the corresponding coefficients from their inner products.

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