

I. INTRODUCTION

对于线性引力波，电磁波，标量波通常可以通过具有有效势的Klein-Gordon (KG)方程来建模：

$$D\phi(x, t) \equiv [\partial_t^2 - \partial_x^2 + V(x)] \phi(x, t) = 0 \quad (\text{I.1})$$

考虑一个静态球对称时空：

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \quad (\text{I.2})$$

Klein-Gordon方程标量场可以表示为：

$$\Phi = \sum_{lm} \frac{1}{r} \phi_{lm}(x, t) Y_{lm}(\theta, \varphi) \quad (\text{I.3})$$

有效势为：

$$V(x) = -g_{tt} \frac{l(l+1)}{r^2} - \frac{1}{2r} \frac{g_{tt}}{g_{rr}} \left[\frac{\partial_r g_{tt}}{g_{tt}} - \frac{\partial_r g_{rr}}{g_{rr}} \right] \quad (\text{I.4})$$

x 与 r 的关系：

$$x = \int \sqrt{-\frac{g_{rr}}{g_{tt}}} dr \quad (\text{I.5})$$

(***这里类似于乌龟坐标 r^* 与 r 的关系 $r^* = \int \frac{1}{f(r)} dr$ ***)

考虑一个史瓦西的时空：

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \quad (\text{I.6})$$

由 (1.5) 可以得到：

$$x = r + 2M \log\left(\frac{r}{2M} - 1\right) \quad (\text{I.7})$$

通过 (1.4) 可以得到势为：

$$V(x) = \left(1 - \frac{2M}{r}\right) \left[\frac{l(l+1)}{r^2} + (1 - S^2) \frac{2M}{r^3} \right] \quad (\text{I.8})$$

(***标量分别对应场对应 $S=0$ ，电磁场和引力场 $S=1, 2$ ***)

II. PART II FORMALISM

(1.1) 中的 $\phi(x, t)$ 可以写为如下形式：

$$\phi(x, t) = \int dy G(x, y; t) \partial_t \phi(y, 0) + \int dy \partial_t G(x, y; t) \phi(y, 0) \quad (\text{II.1})$$

(***

$$\begin{aligned} \nabla_0^2 \psi(r_0, t_0) - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t_0^2} &= -4\pi q(r_0, t_0) \\ \nabla_0^2 G(r, t|r_0, l_0) - \frac{1}{c^2} \frac{\partial^2 G(r, t|r_0, l_0)}{\partial t_0^2} &= -4\pi \delta(r - r_0) \delta(t - t_0) \end{aligned}$$

第一个式子乘 $G(r, t|r_0, l_0)$ 第二个乘 $\psi(r_0, t_0)$ 之后相减再积分：

$$\begin{aligned} &\int_0^{t^+} dt_0 \int dV_0 \left\{ G \nabla_0^2 \psi - \psi \nabla_0^2 G + \frac{1}{c^2} \left(\frac{\partial^2 G}{\partial t_0^2} \psi - G \frac{\partial^2 \psi}{\partial t_0^2} \right) \right\} \\ &= 4\pi \left\{ \psi(r, t) - \int_0^{t^+} dt_0 \int dV_0 q(r_0, t_0) G \right\} \end{aligned}$$

利用高斯定理得到：

$$\begin{aligned} & \int_0^{t^+} dt_0 \oint dS_0 \cdot (G \text{grad}_0 \psi - \psi \text{grad}_0 G) \\ & + \frac{1}{c^2} \int dV_0 \left[\frac{\partial G}{\partial t_0} \psi - G \frac{\partial \psi}{\partial t_0} \right]_0^{t^+} \\ & + 4\pi \int_0^{t^+} dt_0 \int dV_0 q(\mathbf{r}_0, t_0) G = 4\pi \psi(\mathbf{r}, t) \end{aligned}$$

第二项在 $t = t^+$ 时通过 G 的初始条件会消失。所以得到：

$$\begin{aligned} 4\pi \psi(\mathbf{r}, t) &= 4\pi \int_0^{t^+} dt_0 \int dV dG(\mathbf{r}, t | \mathbf{r}_0, t_0) q(\mathbf{r}_0, t_0) \\ &+ \int_0^{t^+} dt_0 \oint dS_0 \cdot (G \text{grad}_0 \psi - \psi \text{grad}_0 G) \\ &- \frac{1}{c^2} \int dV_0 \left[\left(\frac{\partial G}{\partial t_0} \right)_{t_0=0} \psi_0(\mathbf{r}_0, 0) - G_{t_0=0} \frac{\partial \psi(\mathbf{r}_0, t_0)}{\partial t_0} \Big|_{t_0=0} \right] \end{aligned}$$

化简为一维情况：

$$\begin{aligned} \psi(\mathbf{r}, t) &= \int_0^\infty \int_{-\infty}^\infty G(\mathbf{r}, t | \mathbf{r}_0, t_0) q(\mathbf{r}_0, t_0) d\mathbf{r}_0 dt_0 \\ &+ \int_0^\infty [G(\mathbf{r}, t | \mathbf{r}_0, t_0) \psi_{,t_0}(\mathbf{r}_0, t_0) - \psi(\mathbf{r}_0, t_0) G_{,r_0}(\mathbf{r}, t | \mathbf{r}_0, t_0)]_{r_0=-\infty}^{r_0=+\infty} dt_0 \\ &+ \int_{-\infty}^\infty [G(\mathbf{r}, t | \mathbf{r}_0, t_0) \psi_{,t_0}(\mathbf{r}_0, t_0) - \psi(\mathbf{r}_0, t_0) G_{,t_0}(\mathbf{r}, t | \mathbf{r}_0, t_0)]_{t_0=0} d\mathbf{r}_0 \end{aligned}$$

中间那一项等价于：

$$\int_0^\infty \int_{-\infty}^\infty (G \nabla_0^2 \psi - \psi \nabla_0^2 G) d\mathbf{r}_0 dt_0$$

右边第一项涉及 $q(\mathbf{r}_0, t_0)$ 代表源的效应，第二项为表面项从视界和无穷远，代表空间边界的边界条件效应，第三项涉及初始条件。而本文不涉及“源”，空间边界在无穷远处应该趋于0，不应该对问题的解产生任何影响，所以对积分的贡献也应该消失，所以只剩下第三项，这里 $t_0=0$ ，由于 $G(\mathbf{r}, t | \mathbf{r}_0, 0) = G(\mathbf{r}_0, 0 | \mathbf{r}, -t)$ ，所以得到 (2.1) 式***)
对 $t > 0$ ，得到

$$DG(x, y; t) = \delta(t) \delta(x - y) \quad (\text{II.2})$$

$t < 0, G(x, y; t) = 0$ 傅里叶变换：

$$\tilde{G}(x, y; \omega) = \int_0^\infty dt G(x, y; t) e^{i\omega t} \quad (\text{II.3})$$

得到：

$$\tilde{D}(\omega) \tilde{G} \equiv [-\omega^2 - \partial_x^2 + V(x)] \tilde{G}(x, y; \omega) = \delta(x - y) \quad (\text{II.4})$$

(*** 左边：

$$\tilde{D}(\omega) \tilde{G}(x, y; \omega) = \int_{-\infty}^\infty dt [\partial_t^2 - \partial_x^2 + V(x)] G(x, y; t) e^{i\omega t} = [-\omega^2 - \partial_x^2 + V(x)] \int_{-\infty}^\infty dt G(x, y; t) e^{i\omega t}$$

右边：

$$\int_{-\infty}^\infty dt \delta(t) \delta(x - y) e^{i\omega t} = \delta(x - y)$$

***)

对于非奇点的系统，比如振荡的相对论星体产生引力波，它的左边界条件为：

$$\tilde{\phi}(x = 0, \omega) = 0 \quad (\text{II.5})$$

右边界条件为：

$$\tilde{\phi}(x, \omega) \propto e^{i\omega x}, \quad x \rightarrow +\infty \quad (\text{II.6})$$

对于黑洞，x的变化范围是负无穷到正无穷，这样左边界条件趋于负无穷是事件视界，这样边界条件为：

$$\tilde{\phi}(x, \omega) \propto e^{-i\omega x}, \quad x \rightarrow -\infty \quad (\text{II.7})$$

(***随着 $x \rightarrow \infty$ ， $V \rightarrow 0$ 这样得到2.6和2.5b***)

定义 $f(\omega, x)g(\omega, x)$ 是齐次方程 $\tilde{D}(\omega)f(\omega, x) = \tilde{D}(\omega)g(\omega, x) = 0$ 的解， $f(\omega, x)$ 满足左边界条件， $g(\omega, x)$ 满足右边界条件。为了定义，我们采用如下规范化约定：

$$\lim_{x \rightarrow \infty} [e^{-i\omega x} g(\omega, x)] = 1 \quad (\text{II.8})$$

对于半线问题

$$f(\omega, x=0) = 0, f'(\omega, x=0) = 1 \quad (\text{II.9})$$

(*** 我们设这里 f' 在原点非0，否则如果 $x \rightarrow 0, f \sim x^{l+1}$ ，我们采用 $\lim_{x \rightarrow 0} x^{-(l+1)} f(\omega, x) = 1$ 代替 $f'=1$ ***)

对于全线问题

$$\lim_{x \rightarrow -\infty} [e^{i\omega x} f(\omega, x)] = 1$$

这样得到：

$$\tilde{G}(x, y; \omega) = \begin{cases} \frac{f(\omega, x)g(\omega, y)}{W(\omega)}, & x < y \\ \frac{f(\omega, y)g(\omega, x)}{W(\omega)}, & y < x \end{cases} \quad (\text{II.10})$$

这里朗斯基行列式为：

$$W(\omega) = W(g, f) = g\partial_x f - f\partial_x g$$

(***

$$\tilde{D}(\omega)\tilde{G} \equiv [-\omega^2 - \partial_{r^*}^2 + V(r^*)] \tilde{G}(r^*, y; \omega) = \delta(r^* - y)$$

(1)

在 $r^* > y, r^* < y$ 上式为0。所以分别在两区域可以写为

$$[-\omega^2 - \partial_{r^*}^2 + V(r^*)] \phi(r^*, \omega) = 0$$

(2)

对于 (1)， (2)， 两个区域的格林函数可以写为两个解的线性组合：

$$\tilde{G}(r^*, y; \omega) = \begin{cases} c1(y, \omega)f(\omega, r^*) + c2(y, \omega)g(\omega, r^*), & r^* < y \\ d1(y, \omega)f(\omega, r^*) + d2(y, \omega)g(\omega, r^*), & y < r^* \end{cases}$$

(3)

格林函数在y处连续

$$(c1(y, \omega) - d1(y, \omega))f(\omega, y) + (c2(y, \omega) - d2(y, \omega))g(\omega, y) = 0$$

(4)

x从 $y - 0^+$ 到 $y + 0^+$ 积分，积分是无穷小，则连续函数积分为0。

$$\begin{aligned} & \int_{y-0^+}^{y+0^+} (-\partial_{r^*}^2 + V(r^*) - \omega^2) \tilde{G}(r^*, y; \omega) dr^* \\ &= -\left[\frac{d}{dr^*} \tilde{G}(r^*, y; \omega) \right] \Big|_{y-0^+}^{y+0^+} = \int_{y-0^+}^{y+0^+} \delta(r^* - y) dr^* = 1 \end{aligned}$$

(3) 代入上式得到

$$(c1(y, \omega) - d1(y, \omega))f'(\omega, y) + (c2(y, \omega) - d2(y, \omega))g'(\omega, y) = 1$$

(5)

和 (4) 式联立之后得到

$$\begin{aligned} c1(y, \omega) - d1(y, \omega) &= \frac{g(\omega, y)}{Wg(\omega, r^*), f(\omega, r^*)} \\ c2(y, \omega) - d2(y, \omega) &= \frac{-f(\omega, y)}{Wg(\omega, r^*), f(\omega, r^*)} \end{aligned}$$

(6)

将 (6) 带入 (3) 中得到

$$\tilde{G}(r^*, y; \omega) = \begin{cases} \frac{f(\omega, r^*)g(\omega, y)}{Wg(\omega, r^*), f(\omega, r^*)} + d1(y, \omega)f(\omega, r^*) + c2(y, \omega)g(\omega, r^*), & r^* < y \\ \frac{f(\omega, y)g(\omega, r^*)}{Wg(\omega, r^*), f(\omega, r^*)} + d1(y, \omega)f(\omega, r^*) + c2(y, \omega)g(\omega, r^*), & y < r^* \end{cases}$$

(7)

下面考虑边界条件:

$$f(\omega, r^*) = \begin{cases} e^{-i\omega r^*}, & r^* \rightarrow -\infty \\ A_{out}(\omega)e^{i\omega r^*} + A_{in}(\omega)e^{-i\omega r^*}, & r^* \rightarrow \infty \end{cases}$$

$$g(\omega, r^*) = \begin{cases} B_{out}(\omega)e^{i\omega r^*} + B_{in}(\omega)e^{-i\omega r^*}, & r^* \rightarrow -\infty \\ e^{i\omega r^*}, & r^* \rightarrow \infty \end{cases}$$

则 (3) 中

$$\begin{aligned} r^* \rightarrow -\infty, \tilde{\phi}(r^*, \omega) &\propto e^{-i\omega r^*} \\ \tilde{G}(r^*, y; \omega) &= c1(y, \omega)e^{-i\omega r^*} + c2(y, \omega)(B_{out}(\omega)e^{i\omega r^*} + B_{in}(\omega)e^{-i\omega r^*}) \propto e^{-i\omega r^*} \end{aligned}$$

则 $c2(y, \omega) = 0$ 同理

$$\begin{aligned} r^* \rightarrow \infty, \tilde{\phi}(r^*, \omega) &\propto e^{i\omega r^*} \\ \tilde{G}(r^*, y; \omega) &= d1(y, \omega)(A_{out}(\omega)e^{i\omega r^*} + A_{in}(\omega)e^{-i\omega r^*}) + d2(y, \omega)e^{i\omega r^*} \propto e^{i\omega r^*} \end{aligned}$$

则 $d1(y, \omega) = 0$ 所以 (7) 式化简为:

$$\tilde{G}(r^*, y; \omega) = \begin{cases} \frac{f(\omega, r^*)g(\omega, y)}{Wg(\omega, r^*), f(\omega, r^*)}, & r^* < y \\ \frac{f(\omega, y)g(\omega, r^*)}{Wg(\omega, r^*), f(\omega, r^*)}, & y < r^* \end{cases}$$

***)

III. 势下降的比 X^{-2} 的快的情况

由齐次微分方程 $\tilde{D}(\omega)g(\omega, x) = 0$ 和 x 在无穷远的边界条件可以得到精确解

$$g(\omega, x) = e^{i\omega x} - \int_x^\infty dx' \frac{\sin \omega(x-x')}{\omega} V(x') g(\omega, x') \quad (\text{III.1})$$

(*** 粒子会被散射归结为求解 $\omega^2 + \partial_x^2 g(\omega, x) = V(x)g(\omega, x)$ (1) 格林函数满足

$$\omega^2 + \partial_x^2 \tilde{G}(x, y; \omega) = -\delta(x-y)$$

可以得到 $g(\omega, x) = -\int dy \tilde{G}(x, y; \omega) V(y) g(\omega, y)$ 此解并不唯一因为 $g(\omega, x) = g^{(0)}(\omega, x) - \int dy \tilde{G}(x, y; \omega) V(y) g(\omega, y)$ 也满足上式 (1), 其中 $g^{(0)}(\omega, x)$ 是齐次方程 $-\omega^2 - \partial_x^2 g(\omega, x) = 0$ 的解, 因为在无穷远处 $g(\omega, \infty) = e^{i\omega x}$, 所以

$$g(\omega, x) = e^{i\omega x} - \int dy \tilde{G}(x, y; \omega) V(y) g(\omega, y)$$

(2)

下面求格林函数 $\tilde{G}(x, y; \omega)$, 同上个解释将上个解释的 (6) 带入 (3) 中得到

$$\tilde{G}(x, y; \omega) = d1(y, \omega)g_1(\omega, x) + d2(y, \omega)g_1(\omega, x) - \frac{g_2(\omega, y)g_1(\omega, x) - g_1(\omega, y)g_2(\omega, x)}{Wg_1(\omega, x)g_2(\omega, x)}\theta(x - y)$$

$g_1(\omega, x) = e^{i\omega x}, g_2(\omega, x) = e^{-i\omega x}, W(g_1(\omega, x)g_2(\omega, x)) = -2i\omega$, $d1(y, \omega)d1(y, \omega)$ 由初始条件决定 $\tilde{G}(x_0, y; \omega) = 0$, $\left. \frac{d}{dx}\tilde{G}(x, y; \omega) \right|_{x=x_0} = 0$, 这样 $d1(y, \omega) = d1(y, \omega) = 0$, 所以得到: $\tilde{G}(x, y; \omega) = \frac{e^{-i\omega y}e^{i\omega x} - e^{i\omega y}e^{-i\omega x}}{2i\omega} = \frac{\sin[\omega(x-y)]}{\omega}$ 带入

(2) 式即可获得 (3.1) 式***

作为一级近似 (3.1) 右侧 $V(x')g(\omega, x')$ 可以用零级近似 $V(x')e^{i\omega x'}$ 替代作为势散射问题的玻恩一级近似解这样 (3.1) 可以简化为:

$$g(\omega, x) = e^{i\omega x} - I(\omega, x) \quad (III.2)$$

$$I(\omega, x) = \int_x^\infty dx' \frac{\sin \omega(x-x')}{\omega} V(x') e^{i\omega x'} \quad (III.3)$$

如果 $V(x)$ 是有限范围的, 那么 ω 中就不可能有奇点。若势由指数形式衰减比如 $V(x) \propto e^{-\lambda x}$ 带入 (3.3) 就会得到:

$$I(\omega, x) \propto \frac{e^{(-\lambda+i\omega)x}}{\lambda(\lambda-2i\omega)} \quad (III.4)$$

其中在复数轴上 $\omega = -i\lambda/2$ 有一个极点, 如果势衰减的比指数形式还要快, $\lambda \rightarrow \infty$, 极点就会消失, 使得 $g(\omega, x)$ 解析。对于指数衰减的势, 也可以不通过玻恩近似解出 $g(\omega, x)$ 。在任何情况下, 如果奇点远离 $\omega=0$, 它们将不会对晚期行为作出贡献。

下面考虑势下降的速度小于指数情况。

$$V(x) = \frac{K}{x_o^2} \left(\frac{x_o}{x}\right)^\alpha \quad (III.5)$$

考虑 $\alpha > 2$, 系数 K 吸收到参数 x_o 中。当 α 等于整数 n 时得到:

$$I(\omega, x) = e^{-i\omega x} \left\{ \frac{(2i\omega x_o)^{n-2}}{(n-1)!} \left[\gamma - \sum_{m=1}^{n-2} \frac{1}{m} + \log(-2i\omega x) \right] + \frac{1}{n-1} \left(\frac{x_o}{x}\right)^{n-2} \sum_{m=0, m \neq n-2}^\infty \frac{(2i\omega x)^m}{m!(m+2-n)} \right\} \quad (III.6)$$

(***当 $K=1$ 时利用 (4.3) 式得到

$$I(\omega, x) = \int_x^\infty dx' \frac{\sin \omega(x-x')}{\omega} \frac{x_o^{\alpha-2}}{x'^\alpha} e^{-i\omega x'}$$

利用 $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$ 得到:

$$I(\omega, x) = \int_x^\infty dx' \frac{e^{i\omega(x-x')} - e^{-i\omega(x-x')}}{2i\omega} \frac{x_o^{\alpha-2}}{x'^\alpha} e^{-i\omega x'} = \frac{x_o^{\alpha-2} e^{i\omega x}}{2i\omega} \int_x^\infty dx' \frac{1}{x'^\alpha} - \frac{x_o^{\alpha-2} e^{-i\omega x}}{2i\omega} \int_x^\infty dx' \frac{e^{2i\omega x'} x'}{x'^\alpha}$$

(1)

这里

$$\int_x^\infty dx' \frac{1}{x'^\alpha} (u = \omega x', dx' = \frac{1}{\omega} du) \rightarrow \int_{\omega x}^\infty du \omega^\alpha \frac{1}{u^\alpha} \frac{1}{\omega} = \omega^{\alpha-1} \int_{\omega x}^\infty \frac{1}{u^\alpha} du$$

$$\int_x^\infty dx' \frac{e^{2i\omega x'}}{x'^\alpha} (u = \omega x', dx' = \frac{1}{\omega} du) \rightarrow \int_{\omega x}^\infty du \frac{e^{2iu} \omega^\alpha}{u^\alpha} \frac{1}{\omega} = \int_{\omega x}^\infty du \frac{e^{2iu} \omega^{\alpha-1}}{u^\alpha}$$

带入得到 (1) 得到:

$$I(\omega, x) = \frac{(\omega x_o)^{\alpha-2}}{2i} \left\{ \left[\int_{\omega x}^\infty \frac{du}{u^\alpha} \right] e^{i\omega x} - \left[\int_{\omega x}^\infty \frac{du e^{2iu}}{u^\alpha} \right] e^{-i\omega x} \right\}$$

利用分部积分:

$$I(\omega, x) = \frac{(\omega x_o)^{\alpha-2}}{2i} \left\{ \frac{1}{-\alpha+1} (\omega x)^{-\alpha+1} e^{i\omega x} - \left[\int_{\omega x}^\infty du^{-\alpha+1} e^{2iu} \right] \frac{1}{-\alpha+1} e^{-i\omega x} \right\}$$

$$= \frac{(\omega x_o)^{\alpha-2}}{2i} \left\{ \frac{1}{-\alpha+1} (\omega x)^{-\alpha+1} e^{i\omega x} - \left(\frac{1}{-\alpha+1} e^{-i\omega x} (u^{-\alpha+1} e^{2iu}) \right) \Big|_{\omega x}^\infty + \left[\int_{\omega x}^\infty e^{2iu} 2iu^{-\alpha+1} du \right] \frac{1}{-\alpha+1} e^{-i\omega x} \right\}$$

右边第一，二项消掉得到：

$$I(\omega, x) = \frac{(\omega x_0)^{\alpha-2}}{2i} \left[\int_{\omega x}^{\infty} e^{2iu} 2iu^{-\alpha+1} du \right] \frac{1}{-\alpha+1} e^{-i\omega x}$$

$$\frac{(\omega x_0)^{\alpha-2}}{2i} \frac{1}{-\alpha+1} e^{-i\omega x} \left[\int_{-2i\omega x}^{\infty} \frac{2i(-2i)^{\alpha-1}}{p^{\alpha-1}} e^{-p} \left(-\frac{1}{2i}\right) dp \right]$$

$$= -e^{-i\omega x} \frac{(-2i\omega x_0)^{\alpha-2}}{\alpha-1} \left[\int_{-2i\omega x}^{\infty} \frac{1}{p^{\alpha-1}} e^{-p} dp \right]$$

如果 α 是整数

$$\int_{-2i\omega x}^{\infty} \frac{1}{p^{\alpha-1}} e^{-p} dp = \frac{(-1)^{n-1}}{(n-2)!} \left[\gamma - \sum_{m=1}^{n-2} \frac{1}{m} + \log(-2i\omega x) \right] + \frac{(-1)^{n-1}}{(2i\omega x)^{n-2}} \sum_{m=0}^{\infty} \frac{(2i\omega x)^m}{m!(m+2-n)!}$$

γ 是欧拉常数。

如果 α 不是整数

$$\int_{-2i\omega x}^{\infty} \frac{1}{p^{\alpha-1}} e^{-p} dp = \Gamma(2-\alpha) + \frac{(-1)^{\alpha-1}}{(2i\omega x)^{\alpha-2}} \sum_{m=0}^{\infty} \frac{(2i\omega x)^m}{m!(m+2-\alpha)}$$

***)

(3.6) 中的Log函数将引起沿着复数轴的切割，为了描述切割定义：

$$g_{\pm}(-i\sigma, x) = \lim_{\rightarrow 0^+} g(-i\sigma \pm, x)$$

其中 σ 是正实数，这样割线的贡献为：

$$\Delta(\sigma, x) = g_+(-i\sigma, x) - g_-(-i\sigma, x) = -\lim_{\rightarrow 0^+} [I(-i\sigma+, x) - I(-i\sigma-, x)] \quad (\text{III.7})$$

(***因为 (4.2) 式 $g(\omega, x) = e^{i\omega x} - I(\omega, x)$ 中， $e^{i\omega x}$ 是一个连续函数不会影响两边相减后为0***)
通过 (3.2) 和 (3.6) 可以得到切割的强度为：

$$\Delta(\sigma, x) \simeq 2\pi i \frac{(2\sigma x_0)^{n-2}}{(n-1)!} e^{-\sigma x} \quad (\text{III.8})$$

(***多值函数 $w = \ln z = \ln \rho + i\varphi$, w 的幅角有两个值分别对应 $n=0, n=1$ 时 (顺时针绕一圈)

$$w_1 = \ln \rho_1 + i\varphi_1$$

$$w_2 = \ln \rho_1 + i(\varphi_1 - 2\pi)$$

而通过对比 (3.2) 和 (3.6)，除了Log以外，其他都是连续函数所以不影响

$$-\frac{e^{-i(-i\sigma-)x}}{(\alpha-1)} (-2i(-i\sigma-)x_0)^{\alpha-2} \Gamma(2-\alpha)$$

$$(\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \Gamma(x+1) = x\Gamma(x), \sin \pi\alpha = \frac{e^{i\pi\alpha} - e^{-i\pi\alpha}}{2i}, z_1 = 2i\omega x_0) \rightarrow$$

$$e^{-\sigma x} \frac{2\pi i}{\Gamma(\alpha)} (e^{i\pi})^{\alpha-2} \frac{1}{e^{i\pi\alpha} - e^{-i\pi\alpha}} (w_1 - w_2) = e^{-\sigma x} \frac{2\pi i}{\Gamma(\alpha)} (e^{i\pi})^{\alpha} \frac{1}{e^{i\pi\alpha} - e^{-i\pi\alpha}} z_1^{\alpha-2} (1 - e^{-2\pi i(\alpha-2)})$$

$$= e^{-\sigma x} \frac{2\pi i}{\Gamma(\alpha)} (e^{i\pi})^{\alpha} \frac{1}{e^{i\pi\alpha} - e^{-i\pi\alpha}} (2\sigma x_0)^{\alpha-2} (1 - e^{-2\pi i\alpha}) = e^{-\sigma x} \frac{2\pi i}{\Gamma(\alpha)} (2\sigma x_0)^{\alpha-2}$$

***)

如果 α 不是整数

$$I(\omega, x) = \frac{-e^{-i\omega x}}{(\alpha-1)} \left[(-2i\omega x_0)^{\alpha-2} \Gamma(2-\alpha) - \left(\frac{x_0}{x}\right)^{\alpha-2} \sum_{m=0}^{\infty} \frac{(2i\omega x)^m}{m!(m+2-\alpha)} \right] \quad (\text{III.9})$$

其中 $(-2i\omega x_0)^{\alpha-2}$ 会引起切割也是沿着复数轴，切割的强度为：

$$\Delta(\sigma, x) \simeq 2\pi i \frac{(2\sigma x_0)^{\alpha-2}}{\Gamma(\alpha)} e^{-\sigma x} \quad (\text{III.10})$$

(***多值函数 $w = z^\rho$, w 的幅角有两个值分别对应 $n=0, n=1$ 时 (顺时针绕一圈)

$$\begin{aligned} w_1 &= z_1^\rho = |z_1|^\rho e^{i\rho \text{Arg} z_1} \\ w_2 &= |z_1|^\rho e^{i\rho \text{Arg} z_1 - 2\pi i\rho} = z_1^\rho e^{-2\pi i\rho} \end{aligned}$$

而通过对比 (3.2) 和 (3.8) , 除了 $(-2i\omega x_o)^{\alpha-2}$ 以外, 其他都是连续函数所以不影响

$$\begin{aligned} \Delta(\sigma, x) &= -\lim_{\rightarrow 0^+} \left[\frac{-e^{-i(-i\sigma+)x}}{(\alpha-1)} (-2i(-i\sigma+)x_o)^{\alpha-2} \Gamma(2-\alpha) \right. \\ &\quad \left. - \frac{-e^{-i(-i\sigma-)x}}{(\alpha-1)} (-2i(-i\sigma-)x_o)^{\alpha-2} \Gamma(2-\alpha) \right] \\ &= e^{-\sigma x} \frac{2\pi i}{\Gamma(\alpha)} (e^{i\pi})^{\alpha-2} \frac{1}{e^{i\pi\alpha} - e^{-i\pi\alpha}} (w_1 - w_2) \\ &= e^{-\sigma x} \frac{2\pi i}{\Gamma(\alpha)} (e^{i\pi})^\alpha \frac{1}{e^{i\pi\alpha} - e^{-i\pi\alpha}} z_1^{\alpha-2} (1 - e^{-2\pi i(\alpha-2)}) \\ &= e^{-\sigma x} \frac{2\pi i}{\Gamma(\alpha)} (e^{i\pi})^\alpha \frac{1}{e^{i\pi\alpha} - e^{-i\pi\alpha}} (2\sigma x_o)^{\alpha-2} (1 - e^{-2\pi i\alpha}) \\ &= e^{-\sigma x} \frac{2\pi i}{\Gamma(\alpha)} (2\sigma x_o)^{\alpha-2} \end{aligned}$$

***)

当 α 为整数时 (3.10) 可以回到 (3.8) , 所以使用 (3.10) α 为非整数时更为普遍。 (2.8) 中格林函数沿着负的复数轴有切割, 其中一边 $y < x$ 的可以写为:

$$\tilde{G}_\pm(x, y; \omega) = \frac{f(\omega, y)g_\pm(\omega, x)}{W(g_\pm, f)} \quad (\text{III.11})$$

$$\Delta\tilde{G}(x, y; \sigma) \equiv \tilde{G}_+(x, y; \omega) - \tilde{G}_-(x, y; \omega) \Big|_{\omega=-i\sigma} = \frac{W(g_-, g_+)}{W(g_+, f)W(g_-, f)} f(\omega, x)f(\omega, y) \Big|_{\omega=-i\sigma} \quad (\text{III.12})$$

(***

$$\begin{aligned} \tilde{G}_+(x, y; \omega) - \tilde{G}_-(x, y; \omega) &= \frac{f(\omega, y)g_+(\omega, x)}{W(g_+(\omega, x), f(\omega, x))} - \frac{f(\omega, y)g_-(\omega, x)}{W(g_-(\omega, x), f(\omega, x))} \\ &= \frac{f(\omega, y)g_+(\omega, x)(g_-(\omega, x)f'(\omega, x) - f(\omega, x)g'_-(\omega, x))}{W(g_+(\omega, x), f(\omega, x))W(g_-(\omega, x), f(\omega, x))} \\ &\quad - \frac{f(\omega, y)g_-(\omega, x)(g_+(\omega, x)f'(\omega, x) - f(\omega, x)g'_+(\omega, x))}{W(g_+(\omega, x), f(\omega, x))W(g_-(\omega, x), f(\omega, x))} \\ &= \frac{f(\omega, x)f(\omega, y)(g_-(\omega, x)g'_+(\omega, x) - g_+(\omega, x)g'_-(\omega, x))}{W(g_+(\omega, x), f(\omega, x))W(g_-(\omega, x), f(\omega, x))} \\ &= \frac{W(g_-, g_+)}{W(g_+, f)W(g_-, f)} f(\omega, x)f(\omega, y) \end{aligned}$$

***)

结合 (3.2) 和 (3.9) 得到:

$$W(g_-, g_+) \Big|_{\omega=-i\sigma} \simeq -\frac{2\pi i}{x_o} \frac{(2\sigma x_o)^{\alpha-1}}{\Gamma(\alpha)} \quad (\text{III.13})$$

(***

$$\begin{aligned} &= \lim_{\rightarrow 0^+} \left(e^{i(-i\sigma-)x} + \frac{e^{-i(-i\sigma-)x}}{(\alpha-1)} (-2i(-i\sigma-)x_o)^{\alpha-2} \Gamma(2-\alpha) \right) \\ &\quad (i(-i\sigma+)e^{i(-i\sigma+)x} - i(-i\sigma+) \frac{e^{-i(-i\sigma+)x}}{(\alpha-1)} (-2i(-i\sigma+)x_o)^{\alpha-2} \Gamma(2-\alpha)) \\ &\quad - (e^{i(-i\sigma+)x} + \frac{e^{-i(-i\sigma+)x}}{(\alpha-1)} (-2i(-i\sigma+)x_o)^{\alpha-2} \Gamma(2-\alpha)) \\ &\quad (i(-i\sigma-)e^{i(-i\sigma-)x} - i(-i\sigma-) \frac{e^{-i(-i\sigma-)x}}{(\alpha-1)} (-2i(-i\sigma-)x_o)^{\alpha-2} \Gamma(2-\alpha)) \\ &\simeq \frac{2\sigma}{(\alpha-1)} (-2i(-i\sigma-)x_o)^{\alpha-2} \Gamma(2-\alpha) - \frac{2\sigma}{(\alpha-1)} (-2i(-i\sigma+)x_o)^{\alpha-2} \Gamma(2-\alpha) \\ (4.10) \rightarrow &-2\pi i \frac{(2\sigma x_o)^{\alpha-2} 2\sigma}{\Gamma(\alpha)} e^{-\sigma x} = -\frac{2\pi i}{x_o} \frac{(2\sigma x_o)^{\alpha-1}}{\Gamma(\alpha)} \end{aligned}$$

***)

从 (3.13) 可以看出 $W(g_-, g_+)$ 不依赖 x 的大小。

下面考虑半线问题当 $x=0$ 时:

$$W(g_\pm, f) = g_\pm(\omega, x=0) \quad (\text{III.14})$$

(***)

$$W(g_{\pm}, f) = g_{\pm} f' - f g_{\pm}' (f(\omega, x=0) = 0, f'(\omega, x=0) = 1) \rightarrow g_{\pm}(\omega, x=0)$$

***)

这样 (3.14) 和 (3.13) 带入 (3.12) 中得到:

$$\Delta \tilde{G}(x, y; \sigma) \simeq -\frac{2\pi i (2\sigma x_o)^{\alpha-1}}{x_o \Gamma(\alpha)} \frac{f(-i\sigma, x) f(-i\sigma, y)}{g_+(-i\sigma, 0) g_-(-i\sigma, 0)} \quad (\text{III.15})$$

所以切割沿着负复数轴的贡献为:

$$G_L(x, y; t) = -i \int_0^{\infty} \frac{d\sigma}{2\pi} \Delta \tilde{G}(x, y; -i\sigma) e^{-\sigma t} \quad (\text{III.16})$$

(***傅里叶逆变换***)

(3.15) 带入 (3.16) 中得到:

$$G_L(x, y; t) \simeq -\frac{2^{\alpha-1} x_o^{\alpha-2}}{t^{\alpha}} \frac{f(0, x) f(0, y)}{g(0, 0)^2} \quad (\text{III.17})$$

设初始条件为 $\phi(y, 0) = \phi_o(y)$, $\partial_t \phi(y, 0) = \phi_1(y)$, 将 (3.17) 带入 (2.1) 中, 得到:

$$\phi(x, t) \simeq -\frac{2^{\alpha-1} x_o^{\alpha-2}}{t^{\alpha}} \left[\int_0^{\infty} dy f(0, y) \phi_1(y) \right] \frac{f(0, x)}{g(0, 0)^2} \quad (\text{III.18})$$

(***这里仅考虑了 $\int dy G(x, y; t) \partial_t \phi(y, 0)$, 因为 $\int dy \partial_t G(x, y; t) \phi(y, 0)$ 会出现 $t^{-\alpha-1}$, 当 t 趋于无穷表现为次要项***)

IV. 势是 x^{-2} 形式的

$$V(x) = \frac{\nu(\nu+1)}{x^2}, \quad x > x_c \quad (\text{IV.1})$$

其中 $\nu > 0$, 定义 $\rho = \nu + \frac{1}{2}$, 这里 $V(x) \propto \frac{1}{x^2}$, 除非势的系数很小, 否则这里用玻恩近似不准确, 所以第三节的方法这里不适用。然而这种情况下有精确解, 并考虑在正无穷远处的边界条件可以得到:

$$g(\omega, x) = e^{i(\rho+\frac{1}{2})\frac{\pi}{2}} \sqrt{\frac{\pi\omega x}{2}} H_{\rho}^{(1)}(\omega x) \quad (\text{IV.2})$$

(***将 $e^{i(\rho+\frac{1}{2})\frac{\pi}{2}} \sqrt{\frac{\pi\omega x}{2}} g(\omega, x)$ 带入方程 $(\omega^2 + \partial_x^2)g(\omega, x) = V(x)g(\omega, x)$, 中得到 Bessel 方程

$$g''(\omega, x) + \frac{g'(\omega, x)}{x} + (\omega^2 - \frac{(1+2\nu)^2}{4x^2})g(\omega, x) = 0$$

$$k = \omega, r = kx, dx = dr/k \rightarrow g''(\omega, \frac{r}{k}) + \frac{g'(\omega, \frac{r}{k})}{r} + (1 - \frac{(1+2\nu)^2}{4r^2})g(\omega, \frac{r}{k}) = 0$$

该方程的解有两组线性无关的解 $J_{\rho=\frac{1+2\nu}{2}}(z), N_{\rho}(z), H_{\rho}^{(1)}(z), H_{\rho}^{(2)}(z)$, 再考虑在无穷远处的边界条件:

$$H_{\rho}^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z-\rho\pi/2-\pi/4)}$$

$$H_{\rho}^{(2)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{-i(z-\rho\pi/2-\pi/4)}$$

$$J_{\rho}(z) \sim \sqrt{\frac{2}{\pi z}} \cos(z - \rho\pi/2 - \pi/4)$$

$$N_{\rho}(z) \sim \sqrt{\frac{2}{\pi z}} \sin(z - \rho\pi/2 - \pi/4)$$

因为在无穷远处 $g(\omega, \infty) = e^{i\omega x}$, 所以可以确定它的解为 $H_{\rho}^{(1)}(\omega x)$ 并且 (5.2) 前的系数 $e^{i(\rho+\frac{1}{2})\frac{\pi}{2}} \sqrt{\frac{\pi\omega x}{2}}$ 可以通过无穷远边界 $H_{\rho}^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z-\rho\pi/2-\pi/4)}$ 条件确定。

或者这样变换成贝塞尔函数

$$\begin{aligned} \omega^2 + \partial_x^2 g(\omega, x) &= V(x)g(\omega, x) \\ g(\omega, x) &= \sqrt{x}g_a(\omega, x) \rightarrow g''_a(\omega, x) + \frac{1}{x}g'_a(\omega, x) + (\omega^2 - \frac{(\nu+1/2)^2}{x^2})g_a(\omega, x) = 0 \\ r = \omega x &\rightarrow g''_a(\omega, r/\omega) + \frac{1}{r}g'_a(\omega, r/\omega) + (1 - \frac{(\nu+1/2)^2}{r^2})g_a(\omega, r/\omega) = 0 \end{aligned}$$

这是一个 ρ 阶贝塞尔函数***)

其中，第一类Hankel函数 $H_\rho^{(1)}(z) = J_\rho(z) + iN_\rho(z)$

$$\begin{aligned} J_\rho(z) &\sim \frac{1}{\Gamma(\rho+1)} \left(\frac{z}{2}\right)^\rho + \dots \\ N_\rho(z) &\sim -\frac{\Gamma(\rho)}{\pi} \left(\frac{z}{2}\right)^{-\rho} + \dots \end{aligned} \quad (\text{IV.3})$$

(***)

$$\begin{aligned} J_\alpha(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \\ N_\alpha(x) &= \frac{I_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin\alpha\pi} \end{aligned}$$

***)

从(4.3)可以看出，切割沿着负的复数轴：

$$\begin{aligned} g_+(-i\sigma, x) &= \sqrt{\frac{i\pi\sigma x}{2}} e^{i\nu\pi/2} \left[2\cos\rho\pi H_\rho^{(1)}(i\sigma x) + e^{-i\rho\pi} H_\rho^{(2)}(i\sigma x) \right] \\ g_-(-i\sigma, x) &= \sqrt{\frac{i\pi\sigma x}{2}} e^{i\nu\pi/2} \left[e^{-i\rho\pi} H_\rho^{(2)}(i\sigma x) \right] \end{aligned} \quad (\text{IV.4})$$

(***)分别把 $J_\rho(z)N_\rho(z)$ 的第一项带入

$$\begin{aligned} g(\omega, x) &= e^{i(\rho+\frac{1}{2})\frac{\pi}{2}} \sqrt{\frac{\pi\omega x}{2}} H_\rho^{(1)}(\omega x) \\ &= e^{i(\nu+\frac{1}{2}+\frac{1}{2})\frac{\pi}{2}} \sqrt{\frac{\pi\omega x}{2}} \left(\frac{1}{\Gamma(\rho+1)} \left(\frac{\omega x}{2}\right)^\rho - \frac{i\Gamma(\rho)}{\pi} \left(\frac{\omega x}{2}\right)^{-\rho} \right) \\ g_-(-i\sigma, x) &= \lim_{\rightarrow 0^+} (e^{i(\nu+1)\frac{\pi}{2}} \sqrt{\frac{\pi(-i\sigma-x)}{2}} \left(\frac{1}{\Gamma(\rho+1)} \left(\frac{-i\sigma-x}{2}\right)^\rho - \frac{i\Gamma(\rho)}{\pi} \left(\frac{-i\sigma-x}{2}\right)^{-\rho} \right)) \end{aligned}$$

当 $n=0, z = i\sigma x$,则上式等于

$$\begin{aligned} g_-(-i\sigma, x) &= e^{i(\nu+1)\frac{\pi}{2}} \sqrt{\pi} \left(\frac{(e^{-i\pi})^{\rho+\frac{1}{2}}}{\Gamma(\rho+1)} \left(\frac{z}{2}\right)^{\rho+\frac{1}{2}} - \frac{i\Gamma(\rho)}{\pi} \left(\frac{z}{2}\right)^{-\rho+\frac{1}{2}} (e^{i\pi})^{-\rho+\frac{1}{2}} \right) \\ &= ie^{i(\nu)\frac{\pi}{2}} \sqrt{\pi} \left(\frac{-i(e^{-i\pi})^\rho}{\Gamma(\rho+1)} \left(\frac{z}{2}\right)^{\rho+\frac{1}{2}} + \frac{\Gamma(\rho)}{\pi} \left(\frac{z}{2}\right)^{-\rho+\frac{1}{2}} (e^{i\pi})^{-\rho} \right) \\ &= e^{i(\nu)\frac{\pi}{2}} \sqrt{\frac{\pi z}{2}} (e^{-i\pi})^\rho \left(\frac{1}{\Gamma(\rho+1)} \left(\frac{z}{2}\right)^\rho + \frac{i\Gamma(\rho)}{\pi} \left(\frac{z}{2}\right)^{-\rho} \right) \end{aligned}$$

所以得到：

$$g_-(-i\sigma, x) = \sqrt{\frac{i\pi\sigma x}{2}} e^{i\nu\pi/2} \left[e^{-i\rho\pi} H_\rho^{(2)}(i\sigma x) \right]$$

$$\begin{aligned} g_+(-i\sigma, x) &= \lim_{\rightarrow 0^+} (e^{i(\nu+1)\frac{\pi}{2}} \sqrt{\frac{\pi(-i\sigma+x)}{2}} \left(\frac{1}{\Gamma(\rho+1)} \left(\frac{-i\sigma+x}{2}\right)^\rho - \frac{i\Gamma(\rho)}{\pi} \left(\frac{-i\sigma+x}{2}\right)^{-\rho} \right)) \\ &= g_-(-i\sigma, x) + \Delta(\sigma, x) \end{aligned}$$

$$\begin{aligned} \Delta(\sigma, x) &= e^{i(\nu)\frac{\pi}{2}} \sqrt{\pi} \left(\frac{(e^{-i\pi})^\rho}{\Gamma(\rho+1)} \left(\frac{z}{2}\right)^{\rho+1/2} (1 - (e^{2\pi i})^{\rho+1/2}) + \frac{i\Gamma(\rho)(e^{-i\pi})^\rho}{\pi} \left(\frac{z}{2}\right)^{-\rho+1/2} (-1 + (e^{-2\pi i})^{-\rho+1/2}) \right) \\ &= e^{i(\nu)\frac{\pi}{2}} \sqrt{\frac{\pi z}{2}} \left(\frac{1}{\Gamma(\rho+1)} \left(\frac{z}{2}\right)^\rho ((e^{-i\pi})^\rho + (e^{\pi i})^\rho) - \frac{i\Gamma(\rho)}{\pi} \left(\frac{z}{2}\right)^{-\rho} ((e^{-i\pi})^\rho + (e^{\pi i})^\rho) \right) \\ &= e^{i(\nu)\frac{\pi}{2}} \sqrt{\frac{\pi z}{2}} (2\cos(\rho\pi)) \left(\frac{1}{\Gamma(\rho+1)} \left(\frac{z}{2}\right)^\rho - \frac{i\Gamma(\rho)}{\pi} \left(\frac{z}{2}\right)^{-\rho} \right) \end{aligned}$$

$$g_+(-i\sigma, x) = g_-(-i\sigma, x) + \Delta(\sigma, x) = \sqrt{\frac{i\pi\sigma x}{2}} e^{i\nu\pi/2} \left[2\cos\rho\pi H_\rho^{(1)}(i\sigma x) + e^{-i\rho\pi} H_\rho^{(2)}(i\sigma x) \right]$$

***)

将 (4.4) 带入朗斯基行列式计算得到:

$$W(g_-, g_+) |_{\omega=-i\sigma} = -4i\sigma \sin \nu\pi \quad (\text{IV.5})$$

这里当 ν 为整数时 (4.5) 将会消失, 这也就再一次说明了 $V(x) = \frac{l(l+1)}{x^2}$ 代表一个纯的离心势垒, 对应于三维空间的自由传播并且不会引起任何散射。当 $\nu \rightarrow 0$, 势会变得很小, 可以用玻恩近似来求解。另外, 在 (4.5) 的极限 $\nu \rightarrow 0$ 趋于 $-4i\sigma\nu\pi$, 而 (3.13) 中 $\alpha \rightarrow 2$ 时, (3.13) 即 $W(g_-, g_+) |_{\omega=-i\sigma} = -4i\sigma\pi$ (当 $\alpha=2$ 时, (4.5) 变为 $V(x) = \frac{K}{x^2}$, 没有 x_0 再吸收 K , 所以 (3.13) 应该乘 $K=\nu(\nu+1)(\nu \rightarrow 0) \rightarrow \nu$ 所以 (3.13) $W(g_-, g_+) |_{\omega=-i\sigma} \rightarrow -4i\sigma\pi\nu$, 这时候这两个情况得到的结果是一致的。所以 $W(g_-, g_+) \propto \sigma^{n-1}$, 只在 $n=2$ 时成立, 但是大小会消失。对 $\omega = -i\sigma \rightarrow 0$ 时,

$$\begin{aligned} g_+(-i\sigma, x) &\sim -\frac{1}{\sqrt{\pi}} e^{i(3\rho/2+1/4)\pi} \Gamma(\rho) \left(\frac{i\sigma x}{2}\right)^{-\nu} \\ g_-(-i\sigma, x) &\sim \frac{1}{\sqrt{\pi}} e^{i(-\rho/2+1/4)\pi} \Gamma(\rho) \left(\frac{i\sigma x}{2}\right)^{-\nu} \end{aligned} \quad (\text{IV.6})$$

(*** 因为 $\omega = -i\sigma \rightarrow 0$, 所以将第一项带入, 其他项为次要相

$$g_-(-i\sigma, x) = \sqrt{\frac{i\pi\sigma x}{2}} e^{i\nu\pi/2} \left[e^{-i\rho\pi} \frac{1}{\Gamma(\rho+1)} \left(\frac{i\sigma x}{2}\right)^\rho + \frac{i\Gamma(\rho)}{\pi} \left(\frac{i\sigma x}{2}\right)^{-\rho} \right]$$

显然第一项消失, 而 $\rho=\nu+1/2$, 所以得到

$$\begin{aligned} g_-(-i\sigma, x) &= \sqrt{\pi} e^{i\nu\pi/2} \left[e^{-i\rho\pi} \left(\frac{i\Gamma(\rho)}{\pi} \left(\frac{i\sigma x}{2}\right)^{-\nu-1/2+1/2}\right) \right] = \sqrt{\pi} e^{i\nu\pi/2} \left[e^{-i\rho\pi/2} e^{-i\pi(\nu+1/2)/2} \left(\frac{e^{i\pi/2}\Gamma(\rho)}{\pi} \left(\frac{i\sigma x}{2}\right)^{-\nu}\right) \right] \\ &= \sqrt{\pi} e^{i\nu\pi/2} \left[e^{-i\pi\rho/2+i\pi/4} \left(\frac{\Gamma(\rho)}{\pi} \left(\frac{i\sigma x}{2}\right)^{-\nu}\right) \right] \end{aligned}$$

$$\begin{aligned} g_+(-i\sigma, x) &= e^{i(\nu)\frac{\pi}{2}} \sqrt{\frac{\pi i\sigma x}{2}} \left((2\cos(\rho\pi)) \left(\frac{1}{\Gamma(\rho+1)} \left(\frac{i\sigma x}{2}\right)^\rho - \frac{i\Gamma(\rho)}{\pi} \left(\frac{i\sigma x}{2}\right)^{-\rho} \right) \right. \\ &\quad \left. + e^{-i\rho\pi} \frac{1}{\Gamma(\rho+1)} \left(\frac{i\sigma x}{2}\right)^\rho + \frac{i\Gamma(\rho)}{\pi} \left(\frac{i\sigma x}{2}\right)^{-\rho} \right) \end{aligned}$$

其中, 第一项和第三项消失:

$$\begin{aligned} g_+(-i\sigma, x) &= e^{i(\nu)\frac{\pi}{2}} \sqrt{\frac{\pi i\sigma x}{2}} \left((e^{-i\rho\pi} + e^{i\rho\pi} \left(-\frac{i\Gamma(\rho)}{\pi} \left(\frac{i\sigma x}{2}\right)^{-\rho}\right) + e^{-i\rho\pi} \frac{i\Gamma(\rho)}{\pi} \left(\frac{i\sigma x}{2}\right)^{-\rho} \right) \\ (i = e^{i\pi/2}) &\rightarrow -e^{i(\nu)\frac{\pi}{2}} \sqrt{\pi} e^{i\rho\pi} \frac{e^{i\pi/2}\Gamma(\rho)}{\pi} \left(\frac{i\sigma x}{2}\right)^{-\nu-1/2+1/2} = -\frac{\Gamma(\rho)}{\sqrt{\pi}} \left(\frac{i\sigma x}{2}\right)^{-\nu} e^{i\rho\pi} e^{i(\nu+1/2)\frac{\pi}{2}} e^{i\pi/4} \\ &= -\frac{\Gamma(\rho)}{\sqrt{\pi}} \left(\frac{i\sigma x}{2}\right)^{-\nu} e^{i3\rho\pi/2} e^{i\pi/4} \end{aligned}$$

***)

由 (5.6) 可以看出 $g_{\pm} \propto x^{-\nu}$, 所以当 $x = x_c$ 时

$$W(g_{\pm}, f) = \left[f'(x_c) + \frac{\nu}{x_c} f(x_c) \right] g_{\pm}(x_c) \quad (\text{IV.7})$$

(***

$$W(g_{\pm}, f) = g_{\pm} f' - f g_{\pm}' = g_{\pm} f' + \frac{\nu}{x_c^{\nu+1}} f = \left[f'(x_c) + \frac{\nu}{x_c} f(x_c) \right] g_{\pm}(x_c)$$

***)

因为 $f(\omega, x)$ 在中 ω 是解析的, 这就意味着 $f(\omega, x) = \sum_{n=0}^{\infty} f_n(x) \omega^n, f_0(x) \neq 0$, 得到:

$$\lim_{\omega=-i\sigma \rightarrow 0} W(g_+, f) W(g_-, f) = - \left[f_0'(x_c) + \frac{\nu}{x_c} f_0(x_c) \right]^2 \frac{\Gamma(\rho)^2}{\pi} \left(\frac{\sigma x_c}{2}\right)^{-2\nu} \quad (\text{IV.8})$$

(***

$$\begin{aligned} \lim_{\omega=-i\sigma \rightarrow 0} W(g_+, f) W(g_-, f) &= - \left[f_0'(x_c) + \frac{\nu}{x_c} f_0(x_c) \right]^2 \frac{\Gamma(\rho)^2}{\pi} \left(\frac{i\sigma x_c}{2}\right)^{-2\nu} e^{i\nu\pi} \\ (i = e^{i\pi/2}) &\rightarrow - \left[f_0'(x_c) + \frac{\nu}{x_c} f_0(x_c) \right]^2 \frac{\Gamma(\rho)^2}{\pi} \left(\frac{\sigma x_c}{2}\right)^{-2\nu} \end{aligned}$$

***)

利用 (3.12) 式得到:

$$\Delta \tilde{G}(x, y; \sigma) \sim \frac{4i\pi \sin \nu\pi}{[f'_0(x_c) + (\nu/x_c) f_0(x_c)]^2} \left(\frac{x_c}{2}\right)^{2\nu} \frac{\sigma^{2\nu+1}}{\Gamma(\nu + \frac{1}{2})^2} f_0(x) f_0(y) \quad (\text{IV.9})$$

(*** $\rho = \nu + \frac{1}{2}$ ***)

带入 (3.16) 中得到:

$$G_L(x, y; t) \simeq \frac{2 \sin \nu\pi}{[f'_0(x_c) + (\nu/x_c) f_0(x_c)]^2} \frac{\Gamma(2\nu + 2)}{\Gamma(\nu + \frac{1}{2})^2} \left(\frac{x_c}{2}\right)^{2\nu} t^{-(2+2\nu)} f_0(x) f_0(y) \quad (\text{IV.10})$$

在第四节中对于低阶的玻恩近似计算可以知道当 $V(x) \sim \frac{1}{x^2}$, 晚期行为为 $\frac{1}{t^2}$, 然而对于非整数 ν , 晚期行为是 $t^{-(2+2\nu)}$, 只有当 $\nu \ll 1$ 时, 两者会趋于一致, 但是对于 $\nu=O(1)$ 差异性来源于波恩的高阶近似。但是当 ν 变为整数时, (4.5) 为 0, 这样晚期行为依然受指数衰减的 QNM 主导。

晚期行为为:

$$\phi(x, t) \sim \frac{2 \sin(\nu\pi)}{[f'_0(x_c) + \nu f_0(x_c)/x_c]^2} \frac{\Gamma(2\nu + 2)}{\Gamma(\nu + \frac{1}{2})^2} \left(\frac{x_c}{2}\right)^{2\nu} t^{-(2+2\nu)} f_0(x) \int_0^\infty dy f_0(y) \phi_1(y) \quad (\text{IV.11})$$

V. 势是复合的形式的

$$\begin{aligned} V(x) &= \frac{l(l+1)}{x^2} + \bar{V}(x), \quad x > x_c, \quad l = 1, 2, \dots \\ \bar{V}(x) &= \frac{x_o^{\alpha-2}}{x^\alpha} \left[\log\left(\frac{x}{x_o}\right) \right]^\beta \end{aligned} \quad (\text{V.1})$$

对于幂律势 $\beta=0$, 对于 Log 势, $\beta=1$ 。这里我们采取 $\alpha > 2$, α 为非整数。因为是整数时只需 $\alpha \rightarrow n$ 即可, 这里 n 是整数。对于 $l = 1, \beta = 0$, (5.1) 又会回到第三节的情况。这里我们将 $\bar{V}(x)$ 看成是一个微扰, 对于的势 $\frac{l(l+1)}{x^2}$ 可以精确解出:

$$g^{(0)}(\omega, x) \equiv i^{l+1}(\omega x) h_l^{(1)}(\omega x) \quad (\text{V.2})$$

这里 $h_l^{(1)}(\omega x)$ 为球贝塞尔函数。

(***同 (5.2) 的解释:

$$\begin{aligned} \omega^2 + \partial_x^2 g(\omega, x) &= V(x) g(\omega, x) \\ (g(\omega, x) = x g_a(\omega, x)) &\rightarrow x g''_a(\omega, x) + 2g'_a(\omega, x) + (\omega^2 - \frac{l(l+1)}{x^2}) x g_a(\omega, x) = 0 \\ \rightarrow g''_a(\omega, x) + \frac{2}{x} g'_a(\omega, x) &+ (\omega^2 - \frac{l(l+1)}{x^2}) g_a(\omega, x) = 0 \end{aligned}$$

因为 l 是整数, 所以这是一个球贝塞尔方程。通过一些变换可以转化为 $l + 1/2$ 阶贝塞尔方程

$$(r = \omega x, g_a(\omega, x) = \sqrt{\frac{\pi}{2r}} g(\omega, r)) \rightarrow g''(\omega, r) + \frac{1}{r} g'(\omega, r) + (\omega^2 - \frac{(l + 1/2)^2}{r^2}) g(\omega, r) = 0$$

, 所以此方程的解为 $J_{l+1/2}(x), J_{-(l+1/2)}(x), N_{l+1/2}(x), H_{l+1/2}^{(1)}(x), H_{l+1/2}^{(2)}(x)$ 中的任意两个。所以对应的球贝塞尔方程也是下面的任意两个:

$$\begin{aligned} j_l(x) &= \sqrt{\frac{\pi}{2x}} J_{l+1/2}(x), \quad j_{-l}(x) = \sqrt{\frac{\pi}{2x}} J_{-l+1/2}(x) \\ n_l(x) &= \sqrt{\frac{\pi}{2x}} N_{l+1/2}(x) \\ h_l^{(1)}(x) &= \sqrt{\frac{\pi}{2x}} H_{l+1/2}^{(1)}(x) \\ h_l^{(2)}(x) &= \sqrt{\frac{\pi}{2x}} H_{l+1/2}^{(2)}(x) \end{aligned}$$

接下来考虑边界条件当x在无穷远时函数为 $e^{i\omega x}$

$$\begin{aligned} H_\rho^{(1)}(z) &\sim \sqrt{\frac{2}{\pi z}} e^{i(z-\rho\pi/2-\pi/4)} \\ H_\rho^{(2)}(z) &\sim \sqrt{\frac{2}{\pi z}} e^{-i(z-\rho+2\pi-\pi/4)} \\ J_\rho(z) &\sim \sqrt{\frac{2}{\pi z}} \cos(z-\rho\pi/2-\pi/4) \\ N_\rho(z) &\sim \sqrt{\frac{2}{\pi z}} \sin(z-\rho\pi/2-\pi/4) \end{aligned}$$

显然只有 $H_\rho^{(1)}(z)$ 符合边界的形式，这样 e^{iz} 前面的系数为

$$\sqrt{\frac{\pi}{2z}} \sqrt{\frac{2}{\pi z}} e^{i(-\frac{(l+1/2)\pi}{2}-\pi/4)}(z=\omega x) \rightarrow \frac{1}{\omega x} e^{i\pi(-l/2-1/2)}(e^{i\pi/2}=i) \rightarrow \frac{1}{\omega x} i^{-(l+1)}$$

所以 (5.2) 前面的系数为 $i^{l+1}(\omega x)$ ***)

所以运用玻恩近似类比 (3.2) 可以写成:

$$g(\omega, x) \simeq g^{(0)}(\omega, x) - I(\omega, x) \quad (V.3)$$

其中,

$$I(\omega, x) \equiv \int_x^\infty dx' M(x, x'; \omega) \bar{V}(x') g^{(0)}(\omega, x') \quad (V.4)$$

这里的格林函数为:

$$M(x, x'; \omega) = \frac{i\omega}{2} x x' [h_l^{(1)}(\omega x') h_l^{(2)}(\omega x) - h_l^{(1)}(\omega x) h_l^{(2)}(\omega x')] \quad (V.5)$$

(***同 (4.1) 的解释

$$\tilde{G}(x, y; \omega) = d1(y, \omega) g_1(\omega, x) + d2(y, \omega) g_1(\omega, x) - \frac{g_2(\omega, y) g_1(\omega, x) - g_1(\omega, y) g_2(\omega, x)}{W g_1(\omega, x) g_2(\omega, x)} \theta(x - y)$$

$$\begin{aligned} g^{(0)}_1(\omega, x) &= i^{l+1}(\omega x) h_l^{(1)}(\omega x) g^{(0)}_2(\omega, x) = i^{l+1}(\omega x) h_l^{(2)}(\omega x) W g^{(0)}_1(\omega, x) g^{(0)}_2(\omega, x) = 2i\omega e^{i\pi} = 2i\omega i^{2l} \\ g_2(\omega, y) g_1(\omega, x) - g_1(\omega, y) g_2(\omega, x) &= i^{2l+2}(\omega y)(\omega x) (h_l^{(2)}(\omega y) h_l^{(1)}(\omega x) - h_l^{(1)}(\omega y) h_l^{(2)}(\omega x)) \end{aligned}$$

$d1(y, \omega) d1(y, \omega)$ 由初始条件决定

$$\tilde{G}(x_0, y; \omega) = 0, \quad \left. \frac{d}{dx} \tilde{G}(x, y; \omega) \right|_{x=x_0} = 0$$

,这样 $d1(y, \omega) = d1(y, \omega) = 0$, 所以得到: $\tilde{G}(x, y; \omega) = \frac{i\omega x y}{2} [h_l^{(1)}(\omega y) h_l^{(2)}(\omega x) - h_l^{(1)}(\omega x) h_l^{(2)}(\omega y)]$ ***)

考虑幂律形式的 $\bar{V}(x) = \frac{x_0^{\alpha-2}}{x^\alpha}$, 带入 (5.4) 得到:

$$I = \frac{i^l \omega x}{2} [I_1(\omega, x) h_l^{(1)}(\omega x) - I_2(\omega, x) h_l^{(2)}(\omega x)] \quad (V.6)$$

其中,

$$I_1(\omega, x) = (\omega x_0)^{\alpha-2} \int_{\omega x}^\infty du \frac{h_l^{(1)}(u) h_l^{(2)}(u)}{u^{\alpha-2}} \quad (V.7)$$

$$I_2(\omega, x) = (\omega x_0)^{\alpha-2} \int_{\omega x}^\infty du \frac{h_l^{(1)}(u)^2}{u^{\alpha-2}} \quad (V.8)$$

(***)

$$\begin{aligned}
I &= (x_o)^{\alpha-2} \frac{i\omega}{2} x \left[h_l^{(2)}(\omega x) \int_x^\infty h_l^{(1)}(\omega y) y^{l+1}(\omega y) h_l^{(1)}(\omega y) \frac{1}{y^\alpha} dy \right. \\
&\quad \left. - h_l^{(1)}(\omega x) \int_x^\infty h_l^{(1)}(\omega y) y^{l+1}(\omega y) h_l^{(2)}(\omega y) \frac{1}{y^\alpha} dy \right] \\
(u = \omega y) &\rightarrow -\frac{\omega i^l}{2} x \left[(x_o)^{\alpha-2} h_l^{(2)}(\omega x) \int_{\omega x}^\infty h_l^{(1)}(u) \omega h_l^{(1)}(\omega) \frac{\omega^{\alpha-2}}{u^{\alpha-2}} \frac{1}{\omega} du - \right. \\
&\quad \left. (x_o)^{\alpha-2} h_l^{(1)}(\omega x) \int_{\omega x}^\infty h_l^{(1)}(u) \omega h_l^{(2)}(u) \frac{\omega^{\alpha-2}}{u^{\alpha-2}} \frac{1}{\omega} du \right] \\
&= -\frac{\omega i^l}{2} x \left[(\omega x_o)^{\alpha-2} h_l^{(2)}(\omega x) \int_{\omega x}^\infty du \frac{h_l^{(1)2}(u)}{u^{\alpha-2}} - (\omega x_o)^{\alpha-2} h_l^{(1)}(\omega x) \int_{\omega x}^\infty du \frac{h_l^{(1)}(u) h_l^{(2)}(u)}{u^{\alpha-2}} \right]
\end{aligned}$$

***)

其中 $I_1(\omega, x)$ 可以表达为:

$$I_1(\omega, x) = 2^{-2l} (\omega x)^{-2l-1} \left(\frac{x_o}{x} \right)^{\alpha-2} \sum_{m=0}^l a_m(l, \alpha) (\omega x)^{2m} \quad (\text{V.9})$$

$I_2(\omega, x)$ 表达为:

$$I_2(\omega, x) = (-1)^{l+1} 2i C(l, \alpha) \frac{\Gamma(2-\alpha)}{(\alpha-1)} (-2i\omega x_o)^{\alpha-2} + (-1)^l (\omega x)^{-2l-1} \left(\frac{x_o}{x} \right)^{\alpha-2} \sum_{m=0}^{\infty} b_m(l, \alpha) (\omega x)^m \quad (\text{V.10})$$

其中,

$$C(l, \alpha) = \prod_{j=0}^{l-1} \frac{\alpha - 2j - 3}{\alpha + 1 + 2j}, \quad l = 1, 2, \dots \quad (\text{V.11})$$

这里当 $l = 0$ 时, $C(l, \alpha) = 1$

(***对于第一类, 第二类球贝塞尔函数可以写为:

$$h_l^{(1)}(u) = i^{-l-1} u^{-1} e^{iu} \sum_{m=0}^l \frac{(l+m)!}{m!(l-m)!} (-2iu)^{-m}$$

(C1)

$$h_l^{(2)}(u) = i^{l+1} u^{-1} e^{-iu} \sum_{m=0}^l \frac{(l+m)!}{m!(l-m)!} (2iu)^{-m}$$

(C2)

通过 (C1) 和 (C2), $I_1(\omega, x) = (\omega x_o)^{\alpha-2} \int_{\omega x}^\infty du \frac{h_l^{(1)}(u) h_l^{(2)}(u)}{u^{\alpha-2}}$, 可以被估计为:

$$\begin{aligned}
\int_{\omega x}^\infty du \frac{h_l^{(1)}(u) h_l^{(2)}(u)}{u^{\alpha-2}} &= \int_{\omega x}^\infty du (i^{-l-1} u^{-1} e^{iu} i^{l+1} u^{-1} e^{-iu} \frac{1}{u^{\alpha-2}} \sum_{m=0}^l \frac{(l+m)!}{m!(l-m)!} (-2iu)^{-m} \sum_{m=0}^l \frac{(l+m)!}{m!(l-m)!} (2iu)^{-m}) \\
&= \int_{\omega x}^\infty du \left(\frac{1}{u^\alpha} \sum_{m=0}^l \frac{(l+m)!}{m!(l-m)!} (-2iu)^{-m} \sum_{m=0}^l \frac{(l+m)!}{m!(l-m)!} (2iu)^{-m} \right)
\end{aligned}$$

先考虑任意某项:

$$\begin{aligned}
&\int_{\omega x}^\infty du \left(\frac{1}{u^\alpha} \frac{(l+m1)!}{m1!(l-m1)!} (-2iu)^{-m1} \frac{(l+m2)!}{m2!(l-m2)!} (2iu)^{-m2} \right) \\
&= \frac{(l+m1)!}{m1!(l-m1)!} \frac{(l+m2)!}{m2!(l-m2)!} \int_{\omega x}^\infty du \left(\frac{1}{u^\alpha} (-1)^{-m1} (2iu)^{-m2-m1} \right) \\
&= \frac{(-1)^{-m1} (2i)^{-m2-m1} (l+m1)!}{m1!(l-m1)!} \frac{(l+m2)!}{m2!(l-m2)!} \int_{\omega x}^\infty du (u)^{-m2-m1-\alpha} \\
&= \frac{(-1)^{-m1} (2i)^{-m2-m1} (l+m1)!}{m1!(l-m1)!} \frac{(l+m2)!}{m2!(l-m2)!} \frac{1}{-m2-m1-\alpha+1} (u)^{-m2-m1-\alpha+1} \Big|_{\omega x}^\infty \\
(m1 = l - m1, m2 = l - m2) &\rightarrow \frac{(-1)^{-l+m1} (2i)^{-2l+m2+m1} (2l-m1)!}{m1!(l-m1)!} \frac{(2l-m2)!}{m2!(l-m2)!} \frac{-1}{m2+m1-2l-\alpha+1} (\omega x)^{-2l+m2+m1-\alpha+1}
\end{aligned}$$

所以,

$$I_1(\omega, x) = (\omega x_o)^{\alpha-2} \frac{(-1)^{-l+m_1} (2i)^{-2l+m_2+m_1} (2l-m_1)! (2l-m_2)!}{m_1!(l-m_1)! m_2!(l-m_2)!} \frac{-1}{m_2+m_1-2l-\alpha+1} (\omega x)^{-2l+m_2+m_1-\alpha+1}$$

$$((-1)^{-l}(i)^{-2l} = 1) \rightarrow \frac{(2)^{-2l}}{(\omega x)^{2l+1}} \left(\frac{x_o}{x}\right)^{\alpha-2} \frac{(-1)^{m_1} (2i)^{m_2+m_1} (2l-m_1)! (2l-m_2)!}{m_1!(l-m_1)! m_2!(l-m_2)!} \frac{1}{(-m_2-m_1+2l+\alpha-1)} (\omega x)^{m_2+m_1}$$

分别交换 m_1, m_2 的次序再算一下, 最后求和两个相加再除以2得到:

$$I_1(\omega, x) = \frac{2^{-2l}}{(\omega x)^{2l+1}} \left(\frac{x_o}{x}\right)^{\alpha-2} \sum_{m_1, m_2=0}^l \frac{(2i)^{m_1+m_2} [(-1)^{m_1} + (-1)^{m_2}] (2l-m_1)! (2l-m_2)!}{2(2l+\alpha-1-m_1-m_2) (l-m_1)! (l-m_2)! m_1! m_2!} (\omega x)^{m_1+m_2}$$

当 $[(-1)^{m_1} + (-1)^{m_2}]$ 中 m_1, m_2 都不是奇数或者偶数时, 这一项会消失。其中 (ωx) 的幂次会是偶数项所以定义:

$$\sum_{m_1=0}^l \sum_{m_2=0}^l \frac{(2i)^{m_1+m_2} [(-1)^{m_1} + (-1)^{m_2}] (2l-m_1)! (2l-m_2)!}{2(2l+\alpha-1-m_1-m_2) (l-m_1)! (l-m_2)! m_1! m_2!} (\omega x)^{m_1+m_2} \equiv \sum_{m=0}^l a_m(l, \alpha) (\omega x)^{2m}$$

带入上式可以得到 (5.9)。对于 $I_2(\omega, x)$ 的估计会更加复杂一点, 因为对 $I_2(\omega, x) = (\omega x_o)^{\alpha-2} \int_{\omega x}^{\infty} du \frac{h_i^{(1)}(u)^2}{u^{\alpha-2}}$ 中 $h_i^{(1)}(u)$ 的平方带入并不能把求和号前的自变量 u 消掉, 所以会得到类似于这样的积分形式:

$$\mathcal{I}(\beta, z) \equiv \int_z^{\infty} \frac{v(t)}{t^\beta} dt$$

(C3)

这里 $v(t)$ 是解析的, $\lim_{t \rightarrow \infty} v(t)t^n = 0$, 假设 $v(t=0) \neq 0$, 因此积分中 $t=0$ 是奇点。对 $v(t)$ 用Taylor展开可以写为: $\sum_{n=0}^{\infty} c_n t^n$, $t \in [0, \infty)$ 收敛。所以对于 $\beta > 1$, 上式可以写为:

$$\mathcal{I}(\beta, z) = \int_z^{\infty} \frac{v(t) - \sum_{n=0}^{N-1} c_n t^n}{t^\beta} dt + \int_z^{\infty} \frac{\sum_{n=0}^{N-1} c_n t^n}{t^\beta}$$

(C4)

对第二部分积分得到:

$$\mathcal{I}(\beta, z) = \int_0^{\infty} \frac{v(t) - \sum_{n=0}^{N-1} c_n t^n}{t^\beta} dt - \sum_{n=0}^{\infty} \frac{c_n z^{n+1-\beta}}{n+1-\beta}$$

(C5)

实际上这个积分等价于:

$$\mathcal{I}_A(\beta) \equiv \int_0^{\infty} \frac{v(t)}{t^\beta} dt = \frac{e^{\pi\beta i}}{2i \sin \beta\pi} \int_C \frac{v(t)}{t^\beta} dt$$

(C6)

这里 β 是非整数, 并且当 $z \rightarrow 0, z \rightarrow \infty, z^\beta v(t) \rightarrow 0$, 支点在 $z=0, z=\infty$ 。注意这里积分路径是顺时针:

$$\int_0^{\infty} \frac{v(t)}{t^\beta} dt = \int_r^R \frac{v(t)}{t^\beta} dt + \int_{CR} \frac{v(t)}{t^\beta} dt + \int_R^r \frac{v(t)}{t^\beta} e^{-2i\pi\beta} dt + \int_{Cr} \frac{v(t)}{t^\beta} dt$$

$$= \int_C \frac{v(t)}{t^\beta} dt$$

因为 $z \rightarrow 0, z \rightarrow \infty, z^\beta v(t) \rightarrow 0$, 得到

$$\int_{CR} \frac{v(t)}{t^\beta} dt \rightarrow 0 (R \rightarrow \infty)$$

$$\int_{Cr} \frac{v(t)}{t^\beta} dt \rightarrow 0 (r \rightarrow 0)$$

所以上式取极限时 $R \rightarrow \infty, r \rightarrow 0$ 时

$$\int_0^\infty \frac{v(t)}{t^\beta} dt - \int_0^\infty \frac{v(t)}{t^\beta} e^{-2i\pi\beta} dt = 1 - e^{-2i\pi\beta} \int_0^\infty \frac{v(t)}{t^\beta} dt = \int_C \frac{v(t)}{t^\beta} dt$$

$$\int_0^\infty \frac{v(t)}{t^\beta} dt = \frac{1}{1-e^{-2i\pi\beta}} \int_C \frac{v(t)}{t^\beta} dt = \frac{e^{i\pi\beta}}{e^{i\pi\beta}-e^{-i\pi\beta}} \int_C \frac{v(t)}{t^\beta} dt = \frac{e^{i\pi\beta}}{2i\sin(\pi\beta)} \int_C \frac{v(t)}{t^\beta} dt$$

同样使用类似的技巧可以得到：

$$\int_0^\infty \frac{v(t) - \sum_{n=0}^{N-1} c_n t^n}{t^\beta} dt = \frac{e^{\pi\beta i}}{2i \sin \beta\pi} \int_C \frac{v(t) - \sum_{n=0}^{N-1} c_n t^n}{t^\beta} dt$$

(C7)

这里， $\int_C \frac{\sum_{n=0}^{N-1} c_n t^n}{t^\beta} dt$ 会消失因为是解析的绕一圈后为0。所以这里 $\mathcal{I}_A(\beta)$ 可以等价于： $\mathcal{I}_A(\beta) = \int_0^\infty \frac{v(t) - \sum_{n=0}^{c_n t^n}}{t^\beta} dt$
(C8) 这里 (C5) 可以写为：

$$\mathcal{I}(\beta, z) = \mathcal{I}_A(\beta) - \sum_{n=0}^{\infty} \frac{c_n z^{n+1-\beta}}{n+1-\beta}$$

(C9)

(5.8) 可以写为：

$$I_2(\omega, x) = (\omega x_0)^{\alpha-2} J_2(\omega x)$$

$$J_2(\omega x) \equiv \int_{\omega_2}^{\infty} dt \frac{[h_i^{(1)}(t)]^2}{t^{\alpha-2}}$$

(C10-C11)

(C10-C11) 设 $w(t) = [h_l^{(1)}(t)]^2 t^{2l+2}$ (C12)

利用 (C1) 可以写为

$$w(t) = (-1)^{l+1} \sum_{k_1=0}^l \sum_{k_2=0}^l \sum_{n=0}^{\infty} (2i)^{n-k_1-k_2} \times \frac{(l+k_1)!(l+k_2)!(-1)^{k_1+k_2}}{k_1!k_2!(l-k_1)!(l-k_2)!n} t^{n+2l-k_1-k_2} \equiv (-1)^{l+1} \sum_{m=0}^{\infty} d_m t^m$$

(C13) ***

$$w(t) = [h_l^{(1)}(t)]^2 t^{2l+2} = i^{-2l-2} t^{-2} e^{2it} \left(\sum_{m=0}^l \frac{(l+m)!}{m!(l-m)!} (-2it)^{-m} \right)^2 t^{2l+2}$$

对任意的 k_1, k_2 , 上式部分可以写为

$$\frac{(l+m)!}{m!(l-m)!} (-2it)^{-m} = \frac{(l+k_1)!}{k_1!(l-k_1)!} ((-2it)^{-k_1}) \frac{(l+k_2)!}{k_2!(l-k_2)!} ((-2it)^{-k_2})$$

$$= \frac{(l+k_1)!}{k_1!(l-k_1)!} \frac{(l+k_2)!}{k_2!(l-k_2)!} (-1)^{-k_1-k_2} (2i)^{-k_1-k_2} (t)^{-k_1-k_2}$$

再一起求和得到：

$$w(t) = [h_l^{(1)}(t)]^2 t^{2l+2} = i^{-2l-2} t^{-2} e^{2it} \left(\sum_{m=0}^l \frac{(l+m)!}{m!(l-m)!} (-2it)^{-m} \right)^2 t^{2l+2}$$

$$(i^{-2l-2} = (-1)^{l+1}, e^{2it} = \sum_{n=0}^{\infty} \frac{(2it)^n}{n!}, (-1)^{-k_1-k_2} = (-1)^{k_1+k_2}) \rightarrow$$

$$(-1)^{l+1} \sum_{k_1=0}^l \sum_{k_2=0}^l \sum_{n=0}^{\infty} (2i)^{n-k_1-k_2} \times \frac{(l+k_1)!(l+k_2)!(-1)^{k_1+k_2}}{k_1!k_2!(l-k_1)!(l-k_2)!n} t^{n+2l-k_1-k_2}$$

$$J_2(\omega x) = \int_{\omega_2}^{\infty} dt \frac{[h_i^{(1)}(t)]^2}{t^{\alpha-2}} \left([h_l^{(1)}(t)]^2 = w(t)/t^{2l+2} \right) \rightarrow \int_{\omega x}^{\infty} dt \frac{w(t)}{t^{2l+\alpha}}$$

(C14)

这就类似于 (C3) 的形式 $\mathcal{I}(\beta, z) \equiv \int_z^\infty \frac{v(t)}{t^\beta} dt$, 这里 $v(t) = w(t)\beta = 2l + \alpha$, 这样 (C9) 可以写为:

$$J_2(\omega x) = I_A + (-1)^l \sum_{m=0}^{\infty} \frac{d_m(\omega x)^{m+1-2l-\alpha}}{m+1-2l-\alpha}$$

(C15)

这里 I_A 是解析的在 $\alpha \leq -l$ 中, 通过查表可以得到:

$$\mathcal{I}_A = \frac{(-1)^{l+1}(-i)^{\alpha-1} \Gamma\left(\frac{3-\alpha+2l}{2}\right) [\Gamma\left(\frac{2-\alpha}{2}\right)]^2 \Gamma\left(\frac{1-\alpha-2l}{2}\right)}{2^\alpha \pi \Gamma(2-\alpha)}$$

(C16)

再通过 gamma 函数的性质

$$\sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \Gamma(x) = \frac{1}{x(x+1)\cdots(x+n-1)} \Gamma(x+n)$$

做一些变化得到

$$\mathcal{I}_A = (-1)^l (-2i)^{\alpha-1} C(l, \alpha) \frac{\Gamma(2-\alpha)}{(\alpha-1)}$$

(C17) ***

$$\begin{aligned} \sqrt{\pi} \Gamma(2-\alpha) &= 2^{2-\alpha-1} \Gamma\left(\frac{2-\alpha}{2}\right) \Gamma\left(\frac{2-\alpha}{2} + \frac{1}{2}\right) = \Gamma\left(\frac{2-\alpha}{2}\right) \Gamma\left(\frac{2-\alpha}{2} + \frac{1}{2}\right) 2^{1-\alpha} \\ &\rightarrow \Gamma\left(\frac{2-\alpha}{2}\right)^2 = \frac{\pi \Gamma(2-\alpha)^2}{\Gamma\left(\frac{3-\alpha}{2}\right)^2} \frac{1}{2^{2-2\alpha}} \\ \Gamma\left(\frac{3-\alpha+2l}{2}\right) &= \Gamma\left(\frac{3-\alpha}{2} + l\right) = \left(\frac{3-\alpha}{2}\right) \left(\frac{3-\alpha}{2} + 1\right) \cdots \left(\frac{3-\alpha}{2} + l - 1\right) \Gamma\left(\frac{3-\alpha}{2}\right) \\ \Gamma\left(\frac{1-\alpha-2l}{2}\right) &= \Gamma\left(\frac{1-\alpha}{2} - l\right) = \frac{1}{\left(\frac{1-\alpha}{2} - l\right) \left(\frac{1-\alpha}{2} - l + 1\right) \cdots \left(\frac{1-\alpha}{2} - 1\right)} \Gamma\left(\frac{1-\alpha}{2}\right) \\ &\frac{\Gamma\left(\frac{3-\alpha+2l}{2}\right) [\Gamma\left(\frac{2-\alpha}{2}\right)]^2 \Gamma\left(\frac{1-\alpha-2l}{2}\right)}{\Gamma(2-\alpha)} \\ &= \frac{\pi \Gamma(2-\alpha)^2}{\Gamma\left(\frac{3-\alpha}{2}\right)^2} \frac{1}{2^{2-2\alpha}} \Gamma\left(\frac{3-\alpha}{2}\right) \Gamma\left(\frac{1-\alpha}{2}\right) \frac{1}{\Gamma(2-\alpha)} \frac{\left(\frac{3-\alpha}{2}\right) \left(\frac{3-\alpha}{2} + 1\right) \cdots \left(\frac{3-\alpha}{2} + l - 1\right)}{\left(\frac{1-\alpha}{2} - l\right) \left(\frac{1-\alpha}{2} - l + 1\right) \cdots \left(\frac{1-\alpha}{2} - 1\right)} \\ (Cl, \alpha) &= \frac{\left(\frac{3-\alpha}{2}\right) \left(\frac{3-\alpha}{2} + 1\right) \cdots \left(\frac{3-\alpha}{2} + l - 1\right)}{\left(\frac{1-\alpha}{2} - l\right) \left(\frac{1-\alpha}{2} - l + 1\right) \cdots \left(\frac{1-\alpha}{2} - 1\right)}, \Gamma\left(\frac{3-\alpha}{2}\right) = \Gamma\left(1 + \frac{1-\alpha}{2}\right) = \frac{1-\alpha}{2} \Gamma\left(\frac{1-\alpha}{2}\right) \rightarrow \\ \mathcal{I}_A &= \frac{(-1)^{l+1}(-i)^{\alpha-1}}{2^\alpha \pi} \frac{\pi}{2^{2-2\alpha}} Cl, \alpha \Gamma(2-\alpha) \frac{2}{1-\alpha} = (-1)^l (-2i)^{\alpha-1} C(l, \alpha) \frac{\Gamma(2-\alpha)}{(\alpha-1)} \end{aligned}$$

$C(l, \alpha) = \prod_{j=0}^{l-1} \frac{\alpha-2j-3}{\alpha+1+2j}$, $l = 1, 2, \dots$ 即 (6.11) 式, 将 (C15), (C17) 带入 (C10) 中, 做一些简化 $b_m \equiv \frac{d_m}{m+1-2l-\alpha}$ (C18) 得到 (6.10)

$$\begin{aligned} I_2(\omega, x) &= (\omega x_0)^{\alpha-2} J_2(\omega x) = (\omega x_0)^{\alpha-2} (-1)^l (-2i)^{\alpha-1} C(l, \alpha) \frac{\Gamma(2-\alpha)}{(\alpha-1)} + (-1)^l \sum_{m=0}^{\infty} \frac{d_m(\omega x)^{m+1-2l-\alpha}}{m+1-2l-\alpha} \\ (b_m \equiv \frac{d_m}{m+1-2l-\alpha}) &\rightarrow (-1)^{l+1} 2i C(l, \alpha) \frac{\Gamma(2-\alpha)}{(\alpha-1)} (-2i\omega x_0)^{\alpha-2} + (-1)^l (\omega x)^{-2l-1} \left(\frac{x_0}{x}\right)^{\alpha-2} \sum_{m=0}^{\infty} b_m(l, \alpha) (\omega x)^m \end{aligned}$$

***)

从 (6.10) 可以看出 $g(\omega, x)$ 中的 $(-2i\omega x_0)^{\alpha-2}$ 将在负的复数轴上引起切割, 切割强度为:

$$\Delta(\sigma, x) \simeq \pi(-i)^{l+1} (2\sigma x_0)^{\alpha-2} \frac{C(l, \alpha)}{\Gamma(\alpha)} (2\sigma x) h_l^{(2)}(-i\sigma x) \quad (\text{V.12})$$

(***)

$$g(\omega, x) = i^{l+1} (\omega x) h_l^{(1)}(\omega x) - \frac{i^l \omega x}{2} \left[\begin{aligned} &2^{-2l} (\omega x)^{-2l-1} \left(\frac{x_0}{x}\right)^{\alpha-2} \sum_{m=0}^l a_m(l, \alpha) (\omega x)^{2m} h_l^{(1)}(\omega x) \\ &- \left[\begin{aligned} &(-1)^{l+1} 2i C(l, \alpha) \frac{\Gamma(2-\alpha)}{(\alpha-1)} (-2i\omega x_0)^{\alpha-2} + \\ &(-1)^l (\omega x)^{-2l-1} \left(\frac{x_0}{x}\right)^{\alpha-2} \sum_{m=0}^{\infty} b_m(l, \alpha) (\omega x)^m \end{aligned} \right] h_l^{(2)}(\omega x) \end{aligned} \right]$$

这里与第五节不同的是 $h_l^{(1)}(\omega x), h_l^{(1)}(\omega x)$ 不会引起切割因为这里从球Hankel函数表达式 (C1-C2)可以看出。

$$\begin{aligned}\Delta(\sigma, x) &= g_+(-i\sigma, x) - g_-(-i\sigma, x) = -\lim_{\rightarrow 0^+} [I(-i\sigma+, x) - I(-i\sigma-, x)] \\ (h_l^{(2)}(\omega x))^{\frac{i\omega x}{2}} (-1)^{l+1} 2i C(l, \alpha) \frac{\Gamma(2-\alpha)}{(\alpha-1)} (-2i\omega x_o)^{\alpha-2} &= i^{l+1}(\omega x) (-1)^{l+1} C(l, \alpha) \frac{\Gamma(2-\alpha)}{(\alpha-1)} (-2i\omega x_o)^{\alpha-2} h_l^{(2)}(\omega x) \rightarrow \\ &= -\lim_{\rightarrow 0^+} [i^{l+1}((-i\sigma+)x) (-1)^{l+1} C(l, \alpha) \frac{\Gamma(2-\alpha)}{(\alpha-1)} (-2i(-i\sigma+)x_o)^{\alpha-2} h_l^{(2)}((-i\sigma+)x) \\ &\quad - i^{l+1}((-i\sigma-)x) (-1)^{l+1} C(l, \alpha) \frac{\Gamma(2-\alpha)}{(\alpha-1)} (-2i(-i\sigma-)x_o)^{\alpha-2} h_l^{(2)}((-i\sigma-)x)]\end{aligned}$$

这里实际上是和 (3.10) 求法完全相同。

多值函数, 的幅角有两个值分别对应 $n=0, n=1$ 时 (顺时针绕一圈)

$$\begin{aligned}w_1 &= z_1^\rho = |z_1|^\rho e^{i\rho \text{Arg} z_1} \\ w_2 &= |z_1|^\rho e^{i\rho \text{Arg} z_1 + 2\pi i \rho} = z_1^\rho e^{-2\pi i \rho}\end{aligned}$$

$$\begin{aligned}(\Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin \pi z}, \Gamma(x+1) = x\Gamma(x), \sin \pi \alpha = \frac{e^{i\pi\alpha} - e^{-i\pi\alpha}}{2i}, z_1 = 2i\omega\sigma) \rightarrow \\ (-i\sigma)x(-i)^{l+1} \frac{C(l, \alpha)}{\Gamma(\alpha)} 2\pi i (-z_1 x_o)^{\alpha-2} h_l^{(2)}((-i\sigma)x) \frac{1}{e^{i\pi\alpha} - e^{-i\pi\alpha}} (w_1 - w_2) \\ &= (-i\sigma)x(-i)^{l+1} \frac{C(l, \alpha)}{\Gamma(\alpha)} 2\pi i (x_o)^{\alpha-2} (e^{i\pi\alpha})^{\alpha-2} h_l^{(2)}((-i\sigma)x) \frac{1}{e^{i\pi\alpha} - e^{-i\pi\alpha}} z_1^{\alpha-2} (1 - e^{-2\pi i(\alpha-2)}) \\ &= (\sigma)x(-i)^{l+1} \frac{C(l, \alpha)}{\Gamma(\alpha)} 2\pi (x_o)^{\alpha-2} h_l^{(2)}((-i\sigma)x) (e^{i\pi\alpha})^\alpha \frac{1}{e^{i\pi\alpha} - e^{-i\pi\alpha}} z_1^{\alpha-2} (1 - e^{-2\pi i\alpha}) \\ &= (2\sigma x)(-i)^{l+1} \frac{C(l, \alpha)}{\Gamma(\alpha)} \pi (x_o)^{\alpha-2} h_l^{(2)}((-i\sigma)x) e^{-\sigma x} (2\sigma x_o)^{\alpha-2}\end{aligned}$$

***)

$W(g_-, g_+)$ 可以估计为:

$$W(g_-, g_+) |_{\omega=-i\sigma} \simeq W(g^{(0)}, \Delta) = -4\pi i \sigma (2\sigma x_o)^{\alpha-2} \frac{C(l, \alpha)}{\Gamma(\alpha)} \quad (\text{V.13})$$

(***

$$\begin{aligned}W(g_-, g_+) &= g_- g_+' - g_+ g_-' = (g_0(-i\sigma-, x) - I(-i\sigma-, x))(g_0'(-i\sigma+, x) - I'(-i\sigma+, x)) \\ &\quad - (g_0(-i\sigma+, x) - I(-i\sigma+, x))(g_0'(-i\sigma-, x) - I'(-i\sigma-, x)) \\ &\simeq -g_0(-i\sigma-, x) I'(-i\sigma+, x) - g_0'(-i\sigma+, x) I(-i\sigma-, x) + g_0(-i\sigma+, x) I'(-i\sigma-, x) \\ &\quad + g_0'(-i\sigma-, x) I(-i\sigma+, x) \\ W(g^{(0)}, \Delta) &= g_0(-i\sigma, x) (I'(-i\sigma-, x) - I'(-i\sigma+, x)) - (I(-i\sigma-, x) - I(-i\sigma+, x)) g_0'(-i\sigma, x) \\ &= -g_0(-i\sigma, x) I'(-i\sigma+, x) - g_0'(-i\sigma, x) I(-i\sigma-, x) + g_0(-i\sigma, x) I'(-i\sigma-, x) + g_0'(-i\sigma, x) I(-i\sigma+, x) \\ g_0(-i\sigma, x) &= \lim_{\rightarrow 0^+} [g_0(-i\sigma-, x)] = \lim_{\rightarrow 0^+} [g_0(-i\sigma+, x)] \\ g_0'(-i\sigma, x) &= \lim_{\rightarrow 0^+} [g_0'(-i\sigma-, x)] = \lim_{\rightarrow 0^+} [g_0'(-i\sigma+, x)] \\ &\rightarrow W(g_-, g_+) \simeq W(g^{(0)}, \Delta)\end{aligned}$$

$$W(g^{(0)}, \Delta) = -4\pi i \sigma (2\sigma x_o)^{\alpha-2} \frac{C(l, \alpha)}{\Gamma(\alpha)}$$

***)

对于 $n \geq 3$ 的整数 (5.13) 可以写为:

$$W(g_-, g_+) = -4\pi i \sigma (2\sigma x_o)^{n-2} \frac{C(l, n)}{(n-1)!} \quad (\text{V.14})$$

$W(g_\pm, f)$ 用 $W(g^{(0)}, f)$ 做近似估计得到:

$$\lim_{\sigma \rightarrow 0} W(g_+, f) W(g_-, f) = \sigma^{-2l} g_o^2$$

这里, $g_o \equiv \lim_{\omega \rightarrow 0} [(i\omega)^l W(g^0, f)]$ 这是一个非0的常数, 当 $l=0$ 退化为 $g(0, 0)$

(***

$$g_o(l=0) \rightarrow \lim_{\omega \rightarrow 0} [W(g^0, f)] (x=0, \omega=0, f(\omega, 0)=0, f'(\omega, 0)=1) \rightarrow g^0(0, 0) f'(0, 0) - g^{0'}(0, 0), f(0, 0) = g^0(0, 0)$$

***)

因此利用 (3.12) 式得到:

$$\begin{aligned} \Delta\tilde{G}(x, y; \sigma) &\equiv \tilde{G}_+(x, y; \omega) - \tilde{G}_-(x, y; \omega) \Big|_{\omega=-i\sigma} = \frac{W(g_-, g_+)}{W(g_+, f)W(g_-, f)} f(\omega, x) f(\omega, y) \Big|_{\omega=-i\sigma} \\ &= \frac{-4\pi i \sigma (2\sigma x_0)^{\alpha-2} \frac{C(l, \alpha)}{\Gamma(\alpha)}}{\sigma^{-2l} g_0^2} f(-i\sigma, x) f(-i\sigma, y) \simeq -\pi i 2^\alpha x_0^{\alpha-2} \sigma^{2l+\alpha-1} \frac{C(l, \alpha)}{\Gamma(\alpha)} \frac{f(-i\sigma, x) f(-i\sigma, y)}{g_0^2} \end{aligned} \quad (V.15)$$

(5.15) 带入 (3.16) $G_L(x, y; t) = -i \int_0^\infty \frac{d\sigma}{2\pi} \Delta\tilde{G}(x, y; -i\sigma) e^{-\sigma t}$ 中得到:

$$G_L(x, y; t) \simeq -\frac{f(0, x) f(0, y)}{g_0^2} C(l, \alpha) 2^\alpha x_0^{\alpha-2} \frac{\Gamma(2l+\alpha)}{\Gamma(\alpha)} t^{-(2l+\alpha)} \equiv -\frac{f(0, x) f(0, y)}{g_0^2} C(l, \alpha) \frac{F(\alpha)}{t^{2l+\alpha}} \quad (V.16)$$

上式对 α 是整数或非整数都成立。但是 α 是奇数并且小于 $2l+3$, $C(l, \alpha)=0$,

(*** $C(l, \alpha) = \prod_{j=0}^{l-1} \frac{\alpha-2j-3}{\alpha+1+2j}$, $l=1, 2, \dots$ 这是连乘, 要是为0只考虑分子即可分子连乘可以写为 $(\alpha-3)(\alpha-5)(\alpha-7)\dots(\alpha+1-2l)(\alpha-1-2l)$, 只要是奇数并且小于 $2l+3$, 分子都为0***)

这种情况下晚期行为在波恩一阶近似下会消失, 所以会考虑高阶的近似。这意味着在这种情况下, 来自渐近远区域的多次散射是很重要的。我们不讨论高阶计算的细节, 但从量纲分析, 我们期望:

$$G_L(x, y; t) \sim t^{-(2l+\alpha)} \left[A_1 + A_2 \left(\frac{x_0}{t} \right)^{\alpha-2} + A_3 \left(\frac{x_0}{t} \right)^{2(\alpha-2)} + \dots \right] \quad (V.17)$$

这里 A_i 是常数正比于第 i 阶的散射振幅。

对于 $\bar{V}(x) = \frac{x_0^{\alpha-2}}{x^\alpha} \left[\log \left(\frac{x}{x_0} \right) \right]$, 会得到:

$$G_L(x, y; t) = \left\{ C(l, \alpha) F(\alpha) \frac{\log t}{t^{2l+\alpha}} - \frac{\partial}{\partial \alpha} [C(l, \alpha) F(\alpha)] \frac{1}{t^{2l+\alpha}} \right\} \frac{f(0, x) f(0, y)}{g_0^2} \quad (V.18)$$

(***通过对比

$$\begin{aligned} \bar{V}(x) &= \frac{x_0^{\alpha-2}}{x^\alpha} \\ \bar{V}(x) &= \frac{x_0^{\alpha-2}}{x^\alpha} \left[\log \left(\frac{x}{x_0} \right) \right] \end{aligned}$$

发现Log势正是对幂律的势求 $-\frac{\partial}{\partial \alpha}$ 得到因为

$$-\frac{\partial \left(\frac{x_0^{\alpha-2}}{x^\alpha} \right)}{\partial \alpha} = -\left(\frac{x_0^{\alpha-2}}{x^\alpha} \text{Log}(x_0) - \frac{x_0^{\alpha-2}}{x^\alpha} \text{Log}(x) \right) = \frac{x_0^{\alpha-2}}{x^\alpha} \left[\log \left(\frac{x}{x_0} \right) \right]$$

, 这样对 (5.16) 求 $\frac{\partial}{\partial \alpha}$, 就会得到 (5.18) ***)

这里 $t^{-(2l+\alpha)}(c \log t + d)$, 其中 $c \propto C(l, \alpha)$, 当是奇数并且小于 $2l+3$ 时, 这一项将会消失只剩下 $t^{-(2l+\alpha)}d$ 。