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Klein Paradox for the Klein-Gordon Equation

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The behavior of a beam of spin-zero particles incident on a region of large potential increase is examined. The results are compared with those obtained from similar computations employing the Dirac equation. This comparison yields an instructive illustration of the difference between particles and antiparticles in spin zero and spin one-half single-particle theory.

SOON after the construction of the Dirac theory of the electron, Klein¹ discussed the problem of an electron beam incident on a region of large potential increase. If the potential energy reaches a value V greater than the total energy E of the particles plus the rest energy mc^2 , as illustrated in Fig. 1, it is possible for the electrons to appear beyond the potential increase with negative kinetic energy. This phenomenon, known as the "Klein paradox," has been discussed by various authors. The most extensive calculations have been given by Sauter.² He showed that the probability of transmission to the region of negative kinetic energy decreases rapidly with decreasing slope of the potential energy curve, but that the essentials of the result are independent of the details of this curve. A modern discussion of the Klein paradox is given by, for example, Umezawa.³ In terms of field theory, the situation is, of course, not paradoxical.

It appears, however, that the Klein paradox has not been examined for spin zero particles from the point of view of single-particle theory. When carried through in detail, the Klein-Gordon computation proves to be basically different from the Dirac computation. The differences arise from the dissimilar relations that exist between particles and their antiparticles in the two theories. These relations have, of course, been known for a long time. The purpose of these remarks is simply to use the Klein paradox to illustrate the situation. In Part A, the Klein-Gordon calculation will be presented. In Part B, the well-known Dirac equation calculation will be summarized for comparison. The differences

between the two computations will be discussed in Part C.

Both computations will be made in one dimension only, for states of definite energy E . The mass of the particle is given by m , its charge by q , and the scalar potential by ϕ . The vector potential is taken to be zero. The units are chosen such that $\hbar = 1$, $c = 1$.

A. KLEIN PARADOX FOR THE KLEIN-GORDON EQUATION

The equation to be solved for the spatial part $u(x)$ of the wave function is⁴

$$\frac{d^2u}{dx^2} = [m^2 - (E - q\phi)^2]u. \quad (1)$$

In Region I (Fig. 1), $q\phi = 0$, and

$$u_I = Ae^{+ipx} + Be^{-ipx}, \quad p \equiv (E^2 - m^2)^{1/2}. \quad (2)$$

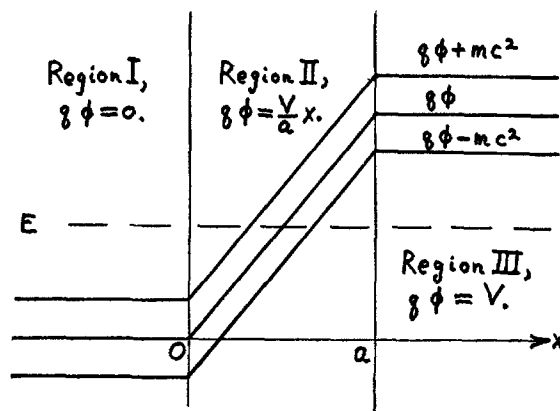


FIG. 1. Form of potential used for examining the Klein paradox.

¹ O. Klein, *Z. Physik* **53**, 157 (1929).

² F. Sauter, *Z. Physik* **69**, 742 (1931); **73**, 547 (1932).

³ H. Umezawa, *Quantum Field Theory* (North-Holland Publishing Company, Amsterdam, 1956).

⁴ For a recent summary of the properties of the Klein-Gordon and Dirac equations and construction of their charge and current densities, see Schweber, Bethe, and de Hoffmann, *Mesons and Fields* (Row, Peterson, and Company, Evanston, 1956), Vol. I, Sec. 2.

The charge density is given by

$$\rho = q \frac{E - q\phi}{m} u^* u, \quad (3)$$

and has the same sign as q in Region I. It is, therefore, proper to identify Ae^{+ipx} with the incident beam and Be^{-ipx} with the reflected beam.

In Region III, $q\phi = V$, and

$$u_{\text{III}} = Ce^{+iP(x-a)} + De^{-iP(x-a)}, \quad (4)$$

$$P \equiv +[(V-E)^2 - m^2]^{\frac{1}{2}}.$$

Since here $q\phi > E$, the sign of the charge density (3) is opposite to that of q . It follows that $De^{-iP(x-a)}$, which yields a current from $x = +\infty$ back to the origin, describes particles of charge $-q$ moving toward $x = +\infty$. On the other hand, $Ce^{+iP(x-a)}$ then describes particles of charge $-q$ moving from $x = +\infty$ toward the origin. If there is to be a source of particles only at $x = -\infty$, C must be set equal to zero.

In Region II, the equation to be solved is

$$\frac{d^2 u_{\text{II}}}{dx^2} = \left[m^2 - \left(E - \frac{V}{a} x \right)^2 \right] u_{\text{II}}. \quad (5)$$

Let

$$s = + (V/a)^{\frac{1}{2}} (x - Ea/V), \quad (6)$$

$$u_{\text{II}} = \exp(-is^2/2) f(s), \quad (7)$$

$$l = m^2 a / V. \quad (8)$$

Then Eq. (5) becomes

$$\frac{d^2 f}{ds^2} - 2is \frac{df}{ds} - (l+i)f = 0. \quad (9)$$

Next, let

$$f(s) = s^r \sum_{n=0}^{\infty} a_n s^n, \quad a_0 \neq 0. \quad (10)$$

Substitution of this expansion then yields the indicial equation

$$r(r-1) = 0, \quad (11)$$

and the recurrence relation

$$a_{n+2} = \frac{2i(r+n) + i + l}{(r+n+2)(r+n+1)} a_n. \quad (12)$$

The two possible values of r , zero and one, then correspond to the two independent solutions in the usual fashion.

Here, only the case of a very steep potential rise will be explored. A step function was not used for the potential only because of the ambiguities regarding the boundary conditions at the step that might arise. For $0 < x < a$,

$$-E(a/V)^{\frac{1}{2}} < s < (V-E)(a/V)^{\frac{1}{2}}. \quad (13)$$

For any E and V , a sufficiently small a will assure that everywhere

$$|s| \ll 1. \quad (14)$$

If E and V are both a few times the rest energy of the particle, then condition (14) is satisfied if the distance a is much smaller than the Compton wavelength of the particle.

For sufficiently small a , then, one can keep only the first term in each of the two independent series for $f(s)$, and replace $\exp(-is^2/2)$ by unity. In terms of x , the general solution in Region II is approximated by simply

$$u_{\text{II}} = F + Gx. \quad (15)$$

A convenient normalization results if the constant D is taken to be one. Then the requirements of continuity of the wave function and its derivative at 0 and at a fix the remaining constants, and the following description of the wave function results:

$$u_{\text{I}} = \frac{1}{2} (1 - P/p + iaP) e^{+ipx} + \frac{1}{2} (1 + P/p + iaP) e^{-ipx},$$

$$u_{\text{II}} = 1 + iaP - iPx, \quad (16)$$

$$u_{\text{III}} = e^{-iP(x-a)}.$$

In the limit $a \rightarrow 0$, the wave function retains continuity and continuity of derivative at the potential increase. It should be noted, incidentally, that, in the two-component representation of the Klein-Gordon equation introduced by Feshbach and Villars,⁵ neither of the two components is continuous at a potential discontinuity.

From the solution (16), it follows that the reflection coefficient is given by

$$R_{\text{KG}} \equiv \frac{|B|^2}{|A|^2} = \frac{(1 + P/p)^2 + (aP)^2}{(1 - P/p)^2 + (aP)^2}, \quad (17)$$

and that the transmission coefficient is given by

$$T_{\text{KG}} \equiv \frac{P|D|^2}{p|A|^2} = \frac{4P/p}{(1 - P/p)^2 + (aP)^2}. \quad (18)$$

⁵ H. Feshbach and F. Villars, *Revs. Modern Phys.* **30**, 24 (1958).

It is important to notice that $R_{KG} > 1$, and that

$$R_{KG} - T_{KG} = 1. \tag{19}$$

Current is, of course, conserved, since negative current moves toward $+\infty$ in Region III. One can picture the process as follows: all particles incident on the potential barrier are reflected. In addition, the barrier is stimulated to emit pairs of particles and antiparticles by the incident beam.

It is of interest to examine the question of energy conservation in this process. In the thought experiment illustrated in Fig. 2, the potential increase occurs between the outer, grounded box and the inner box. An absorber at the end of the inner box absorbs all the particles incident on it. The other walls of both boxes are very thin.

All the particles incident with energy E are reflected with energy E . In addition, more particles with energy E and charge $+q$ join the reflected beam because of the pair creation. But for each such additional particle, another particle of energy E and charge $-q$ appears inside the box and stops in the absorber. In the absorber, it delivers kinetic energy, perhaps as heat, equal to $V - m - E$, and rest energy m . At the same time, to keep the potential of the inner box at V/q , the battery loses an amount of stored energy V . The change in energy of the system consisting of the boxes and the battery is, therefore, $-E$ for every particle of total energy $+E$ that is created to join the reflected beam. Energy is, therefore, conserved if the kinetic energy that the $-q$ particle loses in coming to rest is taken to be positive, and if the rest energy of the $-q$ particle is taken to be positive.

B. KLEIN PARADOX FOR THE DIRAC EQUATION

The equation to be solved for the four-component spatial part $v(x)$ of the wave function is⁴

$$\frac{d}{dx}v = i\alpha_x[(E - q\phi) - m\beta]v. \tag{20}$$

The representation

$$\alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{21}$$

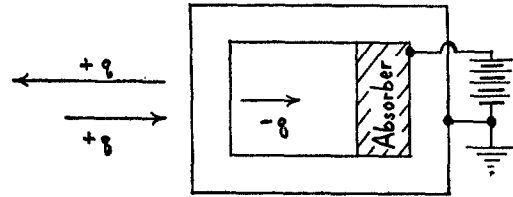


FIG. 2. Thought experiment for the examination of conservation of energy in the Klein paradox.

will be used. That spin state will be taken for which the second and third components are zero. The potential used here is again that illustrated in Fig. 1. The computation will be made as though all states in Region III were available.

In Region I, the solution is

$$v_I = J \begin{pmatrix} 1 \\ 0 \\ 0 \\ \not{p} \\ E+m \end{pmatrix} e^{+ipx+K} + K \begin{pmatrix} 1 \\ 0 \\ 0 \\ -\not{p} \\ E+m \end{pmatrix} e^{-ipx}. \tag{22}$$

The first term describes the incident current moving to the right, and the second term describes the reflected current moving to the left.

In Region III, the general solution is

$$v_{III} = L \begin{pmatrix} -P \\ \frac{V-E+m}{V-E+m} \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+iP(x-a)} + M \begin{pmatrix} +P \\ \frac{V-E+m}{V-E+m} \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{-iP(x-a)}. \tag{23}$$

The Dirac density v^+v , unlike the Klein-Gordon density, has the same sign in Regions I and III.

The current $v^+ \alpha_x v$ must, therefore, be taken positive in Region III and L must equal zero. Notice that both in the Klein-Gordon and the Dirac equation computations, the term containing $e^{-iP(x-a)}$ is retained, but that the corresponding Klein-Gordon current is negative while the Dirac current is positive.

The solution for Region III has been studied in great detail by Sauter.² As mentioned above, the essential features of the problem are illustrated even if a is allowed to approach zero. The correct boundary condition at $x=0$ is then the requirement of continuity of all components, and the following description of the wave function results if $M=1$:

$$v_I = \frac{1}{2} \left(\frac{P}{V-E+m} + \frac{E+m}{p} \right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ p \\ \hline E+m \end{pmatrix} e^{ipx} + \frac{1}{2} \left(\frac{P}{V-E+m} - \frac{E+m}{p} \right) \begin{pmatrix} 1 \\ 0 \\ 0 \\ -p \\ \hline E+m \end{pmatrix} e^{-ipx}. \quad (24a)$$

$$v_{II} = \begin{pmatrix} P \\ \hline V-E+m \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{-iPx}. \quad (24b)$$

The ratio of the magnitude of the reflected current to the incident current gives the reflection coefficient

$$R_D = \frac{[pP - (E+m)(V-E+m)]^2}{[pP + (E+m)(V-E+m)]^2}. \quad (25)$$

The ratio of the transmitted to the incident current gives the transmission coefficient

$$T_D = \frac{4pP(V-E+m)}{[pP + (E+m)(V-E+m)]^2}. \quad (26)$$

Now $R_D < 1$, and

$$R_D + T_D = 1. \quad (27)$$

This result should be compared with Eq. (19).

The energy balance also is different for the Dirac case. Since fewer particles are reflected than incident, an energy $+E$ must be delivered to the system of boxes and battery illustrated in Fig. 2 for every particle that enters the inner box. Since the charge of the transmitted particle is $+q$, an amount of energy V is given to the battery, not taken from it as in the Klein-Gordon case. The kinetic energy lodged in the absorber must then be $-(V-E-m)$, and the rest energy must be $-m$, to give energy conservation.

C. DISCUSSION

The basic difference between the Klein-Gordon and the Dirac computations is brought out by Eqs. (19) and (27). Equation (27) plus the requirement of charge conservation show that the particles in the $E < q\phi$ region have the same charge as the incident particles when the Dirac equation applies. As is well known, it is the hole that results from the removal of a Dirac particle with $E < q\phi$ that must be associated with the antiparticle in discussions of this sort. For the Klein-Gordon case, however, the particles with $E < q\phi$ are themselves the antiparticles that carry charge opposite to that of the particle.

The two discussions of the energy balances illustrate the same facts. A Klein-Gordon negative energy state represents directly an antiparticle that is capable of delivering, in interactions, positive kinetic and rest energies. The corresponding Dirac particle is not described by a negative energy state, but is, of course, described in elementary pictures by a missing negative energy state.