Asymptotic Expansion of Bessel Functions; Applications to Electromagnetics

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Abstract. Bessel function is defined as particular solution of a linear differential equation of the second order known as Bessel's equation. This equation is often used as model of real physical problems. For instance, separation of the wave equation (wave equation in complex domain is called Helmholtz equation) in cylindrical coordinate system leads to Bessel's equation. Besides regular series expansion of the Bessel functions, this paper extends to asymptotic analysis based on contour integral representation of Hankel function.

Keywords: Bessel functions, Asymptotic expansion, Electromagnetics

1 Introduction

Although, there are different approaches, Bessel functions of the first kind are introduced in Section 2 by means of a generating function. In Section 3, Bessel's equation is derived based on field analysis inside a circular waveguide. To define the general system of solutions of the Bessel's equation, we present Bessel functions of the second order known as Neumann functions in Section 4. In section 5, we specify Hankel functions. Section 6 summarize all relations between Bessel functions. Finally, Section 7 explains asymptotic forms of the functions using contour integral definition of Hankel function.

2 Bessel Function of the First Kind, $J_\nu(z)$

One very convenient and instructive way to introduce Bessel functions is due to generating function. This approach provides useful properties of the functions because of its advantage of focusing on the functions themselves rather than on the differential equation they satisfy. Generating function is given in form

$$g(z,t) = e^{(z/2)(t-1/t)} \tag{1}$$

This function is expanded in a Laurent series as function of t and complex variable z:

$$e^{(z/2)(t-1/t)} = \sum_{n=-\infty}^{+\infty} J_n(z) \cdot t^n \tag{2}$$

where $J_n(z)$ are Bessel functions of the first kind, of order n (n is an integer). Expanding the exponentials, we have a product of two absolutely convergent series in $zt/2$ and $-z/(2t)$, respectively:

$$e^{zt/2} \cdot e^{-z/(2t)} = \sum_{r=0}^{\infty} \left(\frac{z}{2}\right)^r \frac{t^r}{r!} \cdot \sum_{m=0}^{\infty} (-1)^m \left(\frac{z}{2}\right)^m \frac{t^{-m}}{m!}$$

$$= \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \left(\frac{z}{2}\right)^{(r+m)} \frac{t^{(r-m)}}{r!m!} \quad (n = r - m,\ r = n + m)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} (-1)^m \left(\frac{z}{2}\right)^{(n+2m)} \frac{t^n}{m!(n+m)!}$$

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!} \left(\frac{z}{2}\right)^{(n+2m)} \tag{3}$$

This is regular expansion for Bessel functions of the first kind, of an integral order n, and it is valid for small argument z ($|z|$). For $n < 0$, Eq.(3) gives:

$$J_{-n}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m-n)!} \left(\frac{z}{2}\right)^{(2m-n)} \tag{4}$$

Because n is an integer, $(m-n)!$ goes to infinity for $m = 0, 1, 2, 3, ...n - 1$ and these terms in the expansion go to zero, so the series may start with $m = n$. Replacing m by $m + n$, we get:

$$J_{-n}(z) = \sum_{m=0}^{\infty} \frac{(-1)^{(n+m)}}{m!(n+m)!} \left(\frac{z}{2}\right)^{(n+2m)} \tag{5}$$

From Eqs. (3) and (5), we conclude that $J_n(z)$ and $J_{-n}(z)$ are not independent but are related by:

$$J_{-n}(z) = (-1)^n J_n(z) \tag{6}$$

This property, Eq. (6), is only valid for an integral order n. These series expansions, Eqs. (3) and (5) are also valid for n replaced by ν to define $J_\nu(z)$ and $J_{-\nu}(z)$ for nonintegral order ν. Bessel functions $J_n(z)$, functions of two variables – unrestricted z and restricted n (integer), are also called Bessel coefficients. General representation of Bessel functions of the first kind, of nonintegral order ν is defined by equation:

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{(\nu+2m)} \tag{7}$$

where Γ is called Gamma function. The Gamma function is an extension of the factorial function with its argument shifted down by one, to real or complex number. If the argument of the function is positive integer:

$$\Gamma(n) = (n-1)! \tag{8}$$

Also, function exists for all complex numbers with a positive real part except non-positive integers and it is defined throughout complex integral as follows:

$$\Gamma(z) = \int_0^{\infty} t^{(z-1)} e^{-t}\, \mathrm{d}t \tag{9}$$

The recurrence formulas for $J_\nu(z)$

$$\begin{aligned} J_{\nu-1}(z) + J_{\nu+1}(z) &= \frac{2\nu}{z} J_\nu(z) \\ J_{\nu-1}(z) - J_{\nu+1}(z) &= 2J'_\nu(z) \\ zJ'_\nu(z) + \nu J_\nu(z) &= zJ_{\nu-1}(z) \\ zJ'_\nu(z) - \nu J_\nu(z) &= -zJ_{\nu+1}(z) \end{aligned} \tag{10}$$

These relations are also valid for integral order n.

3 Cylindrical Waveguide

In this section, we analyze wave propagation along metallic waveguide of circular cross section of radius a; the waveguide is located along the z axis of circular coordinate system (longitudinal axis of the guide coincides with z axis) as indicated in Fig. (1).

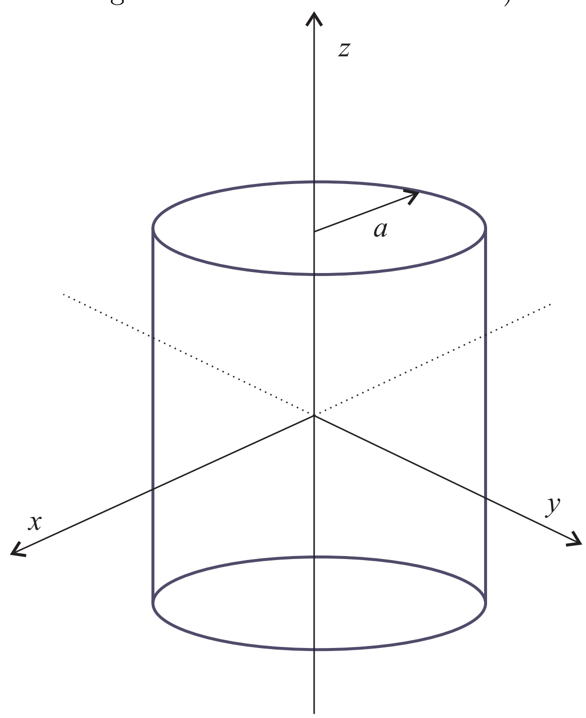

Figure 1: Cylindrical waveguide

We assume that waveguide conductor is perfect, and dielectric inside the guide is homogeneous and without losses. We propose to study electromagnetic field inside the guide. Field expressions for TE and TM waves are derived based on first two Maxwell's equations – coupled system which gives relation between electric and magnetic field:

$$\nabla \times \mathbf{E} = -\mathrm{j}\omega\mu\mathbf{H} \tag{11}$$

$$\nabla \times \mathbf{H} = \mathrm{j}\omega\varepsilon\mathbf{E} \tag{12}$$

where $\mathbf{E}$ is electric field intensity vector, and $\mathbf{H}$ is magnetic field intensity vector [note equations are given in complex domain, so both vectors are complex]; ω is angular frequency, $\omega = 2\pi f$ (f is operating frequency of the guide), ε and μ are relative permittivity and permeability of the guide dielectric , respectively. Here, we show that for TE waves transverse components of electric and magnetic field can be expressed in terms of longitudinal component of magnetic field, and on the other hand, for TM waves, transverse components of electric and magnetic field can be expressed in terms of longitudinal component of electric field. In our case, transverse components are r and ϕ, and z is longitudinal component (in cylindrical coordinate system). Total electric and magnetic field can be written as:

$$\mathbf{E}(r,\phi,z) = E_r(r,\phi,z)\,\hat{\mathbf{r}} + E_\phi(r,\phi,z)\,\hat{\boldsymbol{u\phi}} + E_z(r,\phi,z)\,\hat{\mathbf{z}}$$

$$\mathbf{H}(r,\phi,z) = H_r(r,\phi,z)\,\hat{\mathbf{r}} + H_\phi(r,\phi,z)\,\hat{\boldsymbol{u\phi}} + H_z(r,\phi,z)\,\hat{\mathbf{z}}$$

Assume that wave propagates in positive z direction, therefore, the electric and magnetic field dependance on z coordinate is given by multiplicative factor $e^{-\mathrm{j}\beta z}$, where β is phase coefficient of the wave. Electric field components are expressed as follows:

$$\begin{aligned} E_r(r,\phi,z) &= E_r(r,\phi)e^{-\mathrm{j}\beta z} \\ E_\phi(r,\phi,z) &= E_\phi(r,\phi)e^{-\mathrm{j}\beta z} \\ E_z(r,\phi,z) &= E_z(r,\phi)e^{-\mathrm{j}\beta z} \end{aligned} \tag{13}$$

Magnetic field components are written in the same way. Cross product between ∇ operator and any vector in cylindrical coordinate system $\mathbf{A}$ ($\mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\phi\,\hat{\boldsymbol{u\phi}} + A_z\,\hat{\mathbf{z}}$) is given by following formula:

$$\nabla \times \mathbf{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{\mathbf{r}} + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\hat{\boldsymbol{u\phi}} + \frac{1}{r}\left[\frac{\partial (A_\phi r)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right]\hat{\mathbf{z}} \tag{14}$$

After we apply Eq. (14) to the first Maxwell's equation, Eq. (11), and equal the same vector components from left and right side, we obtain three scalar equations:

$$\frac{1}{r}\left(\frac{\partial E_z}{\partial \phi} + \mathrm{j}\beta r E_\phi\right) = -\mathrm{j}\omega\mu H_r \tag{15}$$

$$\mathrm{j}\beta E_r + \frac{\partial E_z}{\partial r} = \mathrm{j}\omega\mu H_\phi \tag{16}$$

$$\frac{1}{r}\left(\frac{\partial (rE_\phi)}{\partial r} - \frac{\partial E_r}{\partial \phi}\right) = -\mathrm{j}\omega\mu H_z \tag{17}$$

In the similar fashion, we decompose the second Maxwell's equation, Eq. (12) on three scalar equations:

$$\frac{1}{r}\left(\frac{\partial H_z}{\partial \phi} + \mathrm{j}\beta r H_\phi\right) = \mathrm{j}\omega\varepsilon E_r \tag{18}$$

$$\mathrm{j}\beta H_r + \frac{\partial H_z}{\partial r} = -\mathrm{j}\omega\varepsilon E_\phi \tag{19}$$

$$\frac{1}{r}\left(\frac{\partial (rH_\phi)}{\partial r} - \frac{\partial H_r}{\partial \phi}\right) = \mathrm{j}\omega\varepsilon E_z \tag{20}$$

Now, we have system of six scalar equations and six unknown field components. First step is to express transverse electric and magnetic field components, E_r, H_r, E_ϕ, and H_ϕ, in terms of longitudinal components E_z and H_z. If combine Eqs. (16) and (18), we get:

$$E_r(r,\phi) = \frac{-\mathrm{j}}{K^2}\left(\beta\frac{\partial E_z}{\partial r} + \omega\mu\frac{1}{r}\frac{\partial H_z}{\partial \phi}\right) \tag{21}$$

where $K^2 = \omega^2\varepsilon\mu - \beta^2$. Similarly, after combination Eqs. (15) and (19) we obtain radial component of magnetic field expressed due to E_z and H_z:

$$H_r(r,\phi) = \frac{-\mathrm{j}}{K^2}\frac{1}{r}\left(\beta r\frac{\partial H_z}{\partial r} - \omega\varepsilon\frac{\partial E_z}{\partial \phi}\right) \tag{22}$$

Substituting the Eq. (22) into Eq. (19), and Eq. (21) into Eq. (16) we have ϕ component of electric and magnetic field, respectively:

$$E_\phi(r,\phi) = \frac{-\mathrm{j}}{K^2}\left(\frac{\beta}{r}\frac{\partial E_z}{\partial \phi} - \omega\mu\frac{\partial H_z}{\partial r}\right) \tag{23}$$

$$H_\phi(r,\phi) = \frac{-\mathrm{j}}{K^2}\left(\omega\varepsilon\frac{\partial E_z}{\partial r} + \frac{\beta}{r}\frac{\partial H_z}{\partial \phi}\right) \tag{24}$$

Finally, after substitute Eqs. (22) and (24) into Eq. (20), we have wave equation for longitudinal component of electric field:

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} + K^2 E_z = 0 \tag{25}$$

If substitute Eqs. (21) and (23) into Eq. (17), we derive wave equation for longitudinal component of magnetic field vector in the same form as Eq. (25):

$$\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r}\frac{\partial H_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_z}{\partial \phi^2} + K^2 H_z = 0 \tag{26}$$

Second step is to transform, for example, Eq. (25) into Bessel's equation using the method of separation of variables. Let $E_z(r,\phi) = R(r)\Phi(\phi)$ plug into Eq. (25), divide it with product $R(r)\Phi(\phi)$, and then after multiplication with r^2, we get:

$$\frac{r^2}{R}\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{r}{R}\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{1}{\Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}\phi} + r^2K^2 = 0 \tag{27}$$

Note that $\frac{1}{\Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}\phi}$ does not depend on r coordinate, so it must be constant, we take $-n^2$:

$$\frac{1}{\Phi}\frac{\mathrm{d}\Phi}{\mathrm{d}\phi} = -n^2 \tag{28}$$

This equation has general solution in form:

$$\Phi(\phi) = C_1\sin(n\phi) + C_2\cos(n\phi) = C_3\cos(n\phi + \phi_0)$$

It is obvious that field expressions must be periodic in terms of ϕ coordinate, $\Phi(\phi + 2\pi) = \Phi(\phi)$, so constant n must be an integer; constant ϕ_0 can be real and it defines the wave polarization, because of simplicity we take $\phi_0 = 0$. Now, Eq. (27) can be written as:

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r} + (K^2 - \frac{n^2}{r^2})R = 0 \tag{29}$$

This equation represents Bessel's equation in cylindrical coordinate system, and the general solution is given in form:

$$R(r) = \widetilde{C}_1 J_n(Kr) + \widetilde{C}_2 Y_n(Kr)$$

where $J_n(x)$ is Bessel function of first kind, of integral order n, and $Y_n(x)$ is Bessel function of the second kind of order n (known as Neumann function), shown in Figs. (2) and (3), respectively.

Neumann functions tends to infinity if argument tends to zero (they have logarithmic singularity). Because the field must be finite at $r = 0$, therefore $\tilde{C}_2 = 0$. Finally, the expression for z component of electric field intensity vector is:

$$E_z(r, \phi, z) = E_0 J_n(Kr)\cos(n\phi)e^{-\mathrm{j}\beta z} \tag{30}$$

where $E_0 = C_3\widetilde{C}_1$. By analogy, from Eq. (26) we get the expression for z component of magnetic field intensity vector:

$$H_z(r, \phi, z) = H_0 J_n(Kr)\cos(n\phi)e^{-\mathrm{j}\beta z} \tag{31}$$

Substituting Eqs. (30) and (31) into Eqs. (21), (22), (23), and (24), we obtain the expressions for transverse field components, $E_r(r, \phi)$, $H_r(r, \phi)$, $E_\phi(r, \phi)$, and $H_\phi(r, \phi)$, respectively. If $E_z = 0$ and $H_z \neq 0$, we have transverse electric waves, TE waves (electric field has only transverse components, and magnetic field has both, transverse and longitudinal components). If $E_z \neq 0$ and $H_z = 0$, we have transverse magnetic waves, TM waves (magnetic field has only transverse components, and electric field has both, transverse and longitudinal components). Finally, if $E_z \neq 0$ and $H_z \neq 0$, we have hybrid waves (electric and magnetic field have both, transverse and longitudinal components) along the guide.

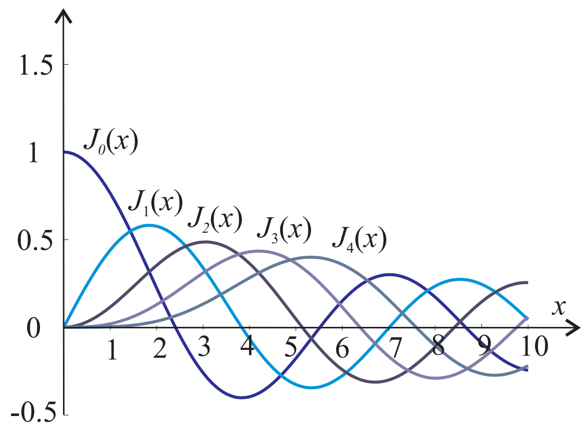

Figure 2: Bessel functions

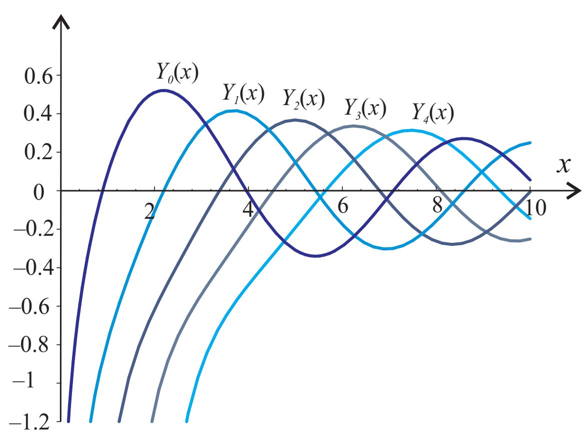

Figure 3: Neumann functions

A fundamental system of solutions of Bessel's equation

Replacing n with ν (noninteger) in Eq. (29), we have more general form of Bessel's equation. Total solution of linear differential equation of second order includes two independent solutions. When ν is not an integer, a fundamental system of solutions of the equation is formed by functions $J_n u(z)$ and $J_{-\nu}(z)$. Otherwise, for $\nu = n$ $J_n(z)$ and $J_{-n}(z)$ are not linearly independent (See Eq. (6)), so they do not form a fundamental system of solutions of Bessel's equation. To determine the fundamental system of solutions, we introduce Neumann functions – Bessel functions of second kind.

4 Neumann Functions, $Y_\nu(z)$

Whenever ν is not an integer, a fundamentals system of solutions of Bessel's equation for functions of order ν is formed by pair $J_\nu(z)$ and $J_{-\nu}$. In case when $\nu = n$ (n is an integer), functions $J_n(z)$ and $J_{-n}(z)$ are linearly dependent, so $J_{-n}(z)$ is not second solution of the equation. The second solution is obtained as combination $J_\nu(z)$ and $J_{-\nu}(z)$ as follows:

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\pi\nu) - J_{-\nu}(z)}{\sin(\pi\nu)} \tag{32}$$

This is Neumann function that also satisfy Bessel's equation because it is liner combination of functions $J_\nu(z)$ and $J_{-\nu}(z)$. When $\nu = n$, the second solution of the equation is given in following form:

$$Y_n(z) = \lim_{\nu\to n} \frac{J_\nu(z)\cos(\pi\nu) - J_{-\nu}(z)}{\sin(\pi\nu)} \tag{33}$$

More general form of Eq. (33) has been given by Neumann:

$$Y_n(z) = J_n(z)\{\log z - s_n\} - \sum_{m=0}^{n-1} \frac{2^{(n-m-1)} n! J_m(z)}{(n-m)! m! z^{(n-m)}}$$

$$+ \sum_{m=1}^{\infty} \frac{(-1)^{(m-1)}(n+2m)}{m(n+m)} J_{n+2m}(z) \tag{34}$$

where $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}, \quad s_0 = 0$

5 Hankel Functions, $H_\nu(z)$

Hankel functions are given as two combinations of Bessel functions of the first and second kind, $J_\nu(z)$ and $Y_\nu(z)$ as follows:

$$H_\nu^{(1)}(z) = J_\nu(z) + \mathrm{j}Y_\nu(z) \tag{35}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - \mathrm{j}Y_\nu(z) \tag{36}$$

where $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ stand for Hankel functions of the first and second kind, respectively. Since $J_\nu(z)$ and $Y_\nu(z)$ satisfy the same recurrence relations, Eq. (10), and since the functions of the third kind are linear combination of J_ν and Y_ν, it follows that these same recurrence formulas are valid for functions of the third kind, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$

6 Relations Between the Three Kind of Bessel Functions

The following formulas express each function in terms of functions of the other two kinds.

$$J_\nu(z) = \frac{H_\nu^{(1)}(z) + H_\nu^{(2)}(z)}{2} = \frac{Y_{-\nu}(z) + Y_\nu(z)\cos(\pi\nu)}{\sin(\pi\nu)} \,, \tag{37}$$

$$J_{-\nu}(z) = \frac{e^{\mathrm{j}\pi\nu} H_\nu^{(1)}(z) + e^{-\mathrm{j}\pi\nu} H_\nu^{(2)}(z)}{2} = \frac{Y_{-\nu}(z)\cos(\pi\nu) - Y_\nu(z)}{\sin(\pi\nu)} \,, \tag{38}$$

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\pi\nu) - J_{-\nu}(z)}{\sin(\pi\nu)} = \frac{H_\nu^{(1)}(z) - H_\nu^{(2)}(z)}{2\mathrm{j}} \,, \tag{39}$$

$$Y_{-\nu}(z) = \frac{J_\nu(z) - J_{-\nu}(z)\cos(\pi\nu)}{\sin(\pi\nu)} = \frac{e^{\mathrm{j}\pi\nu} H_\nu^{(1)}(z) - e^{-\mathrm{j}\pi\nu} H_\nu^{(2)}(z)}{2\mathrm{j}} \,, \tag{40}$$

$$H_\nu^{(1)}(z) = \frac{J_{-\nu}(z) - e^{-\mathrm{j}\pi\nu} J_\nu(z)}{\mathrm{j}\sin(\pi\nu)} = \frac{Y_{-\nu}(z) - e^{-\mathrm{j}\pi\nu} Y_\nu(z)}{\sin(\pi\nu)} \,, \tag{41}$$

$$H_\nu^{(2)}(z) = \frac{e^{\mathrm{j}\pi\nu} J_\nu(z) - J_{-\nu}(z)}{\mathrm{j}\sin(\pi\nu)} = \frac{Y_{-\nu}(z) - e^{\mathrm{j}\pi\nu} Y_\nu(z)}{\sin(\pi\nu)} \,. \tag{42}$$

7 Asymptotic Expansion

In previous sections, varies representations of Bessel functions were shown in form of series expansion of increasing powers of the argument z, multiplied in some cases by $\log z$ (Eq. (34)). These series are valid for numerical computation when z (z^2) is not so large – it is comparable with $4(\nu+1), 4(\nu+2), 4(\nu+3), \ldots$. In that case series converge rapidly. Frequently, in realistic physical problems it is necessary to know the behaviour of Bessel functions for large values of the argument z ($|z|$) which involves the asymptotic analysis. Regular series expansions of Bessel functions converge slowly. The main goal of this section is to present the new formulas, obtained by asymptotic analysis, as fundamental system of solutions of Bessel's equation for large values of argument z. Here, asymptotic analysis of Bessel functions is introduced based on contour integral representation of Hankel functions.

7.1 Asymptotic expansion of Hankel functions $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$

In this part, we derive the asymptotic expansions of Hankel functions starting with following integral form of Hankel function of the first kind, given by Hankel:

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \frac{e^{\mathrm{j}(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi)}}{\Gamma(\nu + \frac{1}{2})} \int_0^{\infty e^{\mathrm{j}\beta}} e^{-u} u^{\nu - \frac{1}{2}} \left(1 + \frac{\mathrm{j}u}{2z}\right)^{\nu - \frac{1}{2}} \mathrm{d}u \tag{43}$$

Integral is valid when $-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$ and $-\frac{1}{2}\pi + \beta < \arg z < \frac{3}{2}\pi + \beta$, provided that $R(\nu + \frac{1}{2}) > 0$. Now, we apply the binomial theorem to factor $\left(1 + \frac{\mathrm{j}u}{2z}\right)^{\nu - \frac{1}{2}}$. Since the expansion is not convergent all along the path of integration, we take finite number of terms and remainder:

$$\left(1 + \frac{\mathrm{j}u}{2z}\right)^{\nu - \frac{1}{2}} = \sum_{m=0}^{p-1} \frac{(\frac{1}{2} - \nu)_m}{m!} \left(\frac{u}{2\mathrm{j}z}\right)^m + \frac{(\frac{1}{2} - \nu)_p}{(p-1)!} \left(\frac{u}{2\mathrm{j}z}\right)^p \int_0^1 (1-t)^{p-1} \left(1 - \frac{ut}{2\mathrm{j}z}\right)^{(\nu - p - \frac{1}{2})} \mathrm{d}t \tag{44}$$

Then, choose any positive angle δ that satisfies inequalities:

$$|\beta| \le \frac{1}{2}\pi - \delta, \quad |\arg z - (\frac{1}{2}\pi + \beta)| \le \pi - \delta.$$

Therefore, $\arg z$ is bounded as:

$$-\pi + 2\delta \le \arg z \le 2\pi - 2\delta$$

Then,

$$\left|1 - \frac{ut}{2\mathrm{j}z}\right| \ge \sin\delta, \quad \left|\arg\left(1 - \frac{ut}{2\mathrm{j}z}\right)\right| < \pi$$

for the values of t and u under consideration, and so

$$\left|\left(1 - \frac{ut}{2\mathrm{j}z}\right)^{(\nu - p - \frac{1}{2})}\right| \le e^{\pi}(\sin\delta)^{R(\nu - p - \frac{1}{2})} = A_p$$

where A_p is independent of z. After substituting Eq. (44) into Eq. (43) and integrating term-by-term, we find that

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \left[\sum_{m=0}^{p-1} \frac{(\frac{1}{2} - \nu)_m \Gamma(\nu + m + \frac{1}{2})}{m!\Gamma(\nu + \frac{1}{2})(2\mathrm{j}z)^m} + R_p^{(1)}\right], \tag{45}$$

where

$$|R_p^{(1)}| \le \frac{A_p}{(p-1)!}\left|\frac{(\frac{1}{2} - \nu)_p}{\Gamma(\nu + \frac{1}{2})(2\mathrm{j}z)^p}\left\{\int_0^1 (1-t)^{p-1}\,\mathrm{d}t \int_0^{\infty e^{\mathrm{j}\beta}} e^{-u}|u^{\nu + p - \frac{1}{2}}\,\mathrm{d}u|\right\}\right| = B_p|z|^{-p}, \tag{46}$$

where B_p is function of ν, p and δ; it does not depend on z. Now, Eq. (45) can be written as:

$$H_\nu^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{\mathrm{j}(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \left[\sum_{m=0}^{p-1} \frac{(\frac{1}{2} - \nu)_m \Gamma(\nu + m + \frac{1}{2})}{m!\Gamma(\nu + \frac{1}{2})(2\mathrm{j}z)^m} + O(z^{-p})\right] \tag{47}$$

In a similar manner, applying the integral form of Hankel function of the second kind:

$$H_\nu^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \frac{e^{-\mathrm{j}(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi)}}{\Gamma(\nu + \frac{1}{2})} \int_0^{\infty e^{\mathrm{j}\beta}} e^{-u} u^{\nu - \frac{1}{2}} \left(1 - \frac{\mathrm{j}u}{2z}\right)^{\nu - \frac{1}{2}} \mathrm{d}u \tag{48}$$

and changing the sign of imaginary unit j throughout the previous work, we obtain the asymptotic expansion of Hankel function of second kind

$$H_\nu^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-\mathrm{j}(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \left[\sum_{m=0}^{p-1} \frac{(\frac{1}{2} - \nu)_m \Gamma(\nu + m + \frac{1}{2})}{m!\Gamma(\nu + \frac{1}{2})(-2\mathrm{j}z)^m} + O(z^{-p})\right] \tag{49}$$

Extending the asymptotic expansions Eq. (47) and Eq. (49) to infinity, we have

$$H_\nu^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{\mathrm{j}(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \sum_{m=0}^{\infty} \frac{(\frac{1}{2} - \nu)_m \Gamma(\nu + m + \frac{1}{2})}{m!\Gamma(\nu + \frac{1}{2})(2\mathrm{j}z)^m} \tag{50}$$

$$H_\nu^{(2)}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} e^{-\mathrm{j}(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \sum_{m=0}^{\infty} \frac{(\frac{1}{2} - \nu)_m \Gamma(\nu + m + \frac{1}{2})}{m!\Gamma(\nu + \frac{1}{2})(-2\mathrm{j}z)^m} \tag{51}$$

7.2 Asymptotic expansion of $J_\nu(z)$, $J_{-\nu}(z)$, $Y_\nu(z)$, and $Y_{-\nu}(z)$

Combining Eqs. (50) and (51) with Eqs. (37) - (42), we get the asymptotic expansion of Bessel functions of first and second kind, respectively

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left[\cos(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi) \sum_{m=0}^{\infty} \frac{(-1)^m(\nu, 2m)}{(2m)^{2m}} - \sin(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi) \sum_{m=0}^{\infty} \frac{(-1)^m(\nu, 2m+1)}{(2z)^{2m+1}}\right] ,$$

$$J_{-\nu}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left[\cos(z + \frac{1}{2}\pi\nu - \frac{1}{4}\pi) \sum_{m=0}^{\infty} \frac{(-1)^m(\nu, 2m)}{(2m)^{2m}} - \sin(z + \frac{1}{2}\pi\nu - \frac{1}{4}\pi) \sum_{m=0}^{\infty} \frac{(-1)^m(\nu, 2m+1)}{(2z)^{2m+1}}\right] ,$$

$$Y_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left[\sin(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi) \sum_{m=0}^{\infty} \frac{(-1)^m(\nu, 2m)}{(2m)^{2m}} + \cos(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi) \sum_{m=0}^{\infty} \frac{(-1)^m(\nu, 2m+1)}{(2z)^{2m+1}}\right] ,$$

$$Y_{-\nu}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left[\sin(z + \frac{1}{2}\pi\nu - \frac{1}{4}\pi) \sum_{m=0}^{\infty} \frac{(-1)^m(\nu, 2m)}{(2m)^{2m}} + \cos(z + \frac{1}{2}\pi\nu - \frac{1}{4}\pi) \sum_{m=0}^{\infty} \frac{(-1)^m(\nu, 2m+1)}{(2z)^{2m+1}}\right] .$$

where

$$(\nu, m) = (-1)^m \frac{(\frac{1}{2} - \nu)_m (\frac{1}{2} + \nu)_m}{m!} = \frac{\Gamma(\nu + m + \frac{1}{2})}{m!\Gamma(\nu - m + \frac{1}{2})}$$

These asymptotic expansions for Bessel functions are valid for large argument $|z|$ and $|\arg z| < \pi$, and the error due to stopping the summation at any term is the order of magnitude of that term multiplied by $1/z$.

References

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