

# The Klein Paradox and Superradiance

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The classic Klein paradox, particle creation due to the presence of very strong electromagnetic potentials, is reconsidered. Vacuum expectation values of the current and momentum operators are calculated for both massive spin zero and massive spin one-half fields. These new results, used as a measure of the number of particles produced, are compared with the results of others for the vacuum expectation value of the number operator. The relationship between superradiance and pair creation is explained and common conceptual errors are pointed out. © 1988 Academic Press, Inc.

## I. INTRODUCTION

The study of fields in strong electromagnetic backgrounds, known as the Klein paradox in the literature, encompasses two distinct phenomena—superradiance and pair creation. Two recent developments have prompted a resurgence in the study of such fields: first, the possibility of experimental verification of pair creation by such potentials during the scattering of heavy ions [1]; and second, the study of fields in strong gravitational backgrounds [2] which would benefit from a simple qualitative analogue.

The fields considered here are a massive, charged, spin 0 boson field and a massive, charged, spin  $\frac{1}{2}$  fermion field, minimally coupled to a vector potential  $A^\mu$  of the form  $A^\mu = (\Phi, 0, 0, 0)$  where  $\Phi$  is a function only of the cartesian  $x$ -coordinate which goes to  $-V/2$  as  $x$  goes to negative infinity and to  $V/2$  as  $x$  goes to positive infinity. Superradiance and/or pair creation can occur only if  $V$  is greater than twice the rest mass of the boson or fermion (the only case considered in this paper).

Superradiance is said to occur if a single mode incident on the potential barrier has a reflected current greater than its incoming current. To discuss pair creation the relevant field is second quantized in the canonical way; in- and out-states are defined. Pair creation is said to occur if the in-vacuum expectation value of the current operator is nonzero or if the in-vacuum expectation value of the out-number operator is nonzero.

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In Section II we discuss the properties of one complete set of solutions of the dynamical equations, called the in-modes. Wronskian relations for these modes are used to show that superradiance occurs in the boson but not the fermion case. While the analytical results of this section are not new, an early misinterpretation is still prevalent in the literature. At the end of Section II this misinterpretation is corrected and the work of other authors is discussed. In Section III we discuss second quantization. The vacuum expectation values of the current and momentum operators are calculated. The results show that although superradiance occurs only in the boson case, strong electromagnetic potentials can create pairs of either bosons or fermions. The results of this section are believed to be new. In Section IV a second set of solutions of the dynamical equations, called out-modes, is defined. Bogoliubov relations, expressing the out-modes as linear combinations of the in-modes, are calculated. These results are used to compute the vacuum expectation value of the out-number operator in the in-vacuum, i.e., the number of particles produced by the potential at late times in a state which begins as the vacuum at early times. The Bogoliubov calculation is the approach taken by many of the authors listed at the end of Section II. The results confirm pair creation in both the boson and the fermion cases. Appendix 1 records calculations of reflection and transmission coefficients for several specific potentials of the general type considered in the body of the paper. These coefficients have been calculated by many of the authors listed at the end of Section II. Appendix 2 is used to describe properties of the spinors used in the fermion calculation.

## II. MODES AND SUPERRADIANCE

### *Bosons*

The action for a massive scalar field  $\phi$  minimally coupled to an electromagnetic potential  $A^\mu$  is

$$S = - \int d^4x (\phi^* \cdot_\mu \phi \cdot^\mu + m^2 \phi^* \phi), \quad (1)$$

where we use the metric  $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$  and  $\cdot$  is defined such that

$$\begin{aligned} \phi \cdot_\mu &= (\partial_\mu - ieA_\mu)\phi \\ \phi^* \cdot_\mu &= (\partial_\mu + ieA_\mu)\phi^*. \end{aligned}$$

Variation with respect to  $\phi^*$  leads to the dynamical equation

$$\phi \cdot_\mu \cdot^\mu - m^2 \phi = 0. \quad (2)$$

This paper will consider potentials of the form  $A^\mu = (\Phi, 0, 0, 0)$  where  $\Phi$  is a barrier varying only in the  $x$ -direction which goes to  $-V/2$  as  $x$  goes to negative infinity

and to  $V/2$  as  $x$  goes to positive infinity (henceforth denoted as “to the left” and “to the right,” respectively). See Fig. 1. For potentials of this type, choose solutions of the form

$$w_{in} = Ne^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} f(x),$$

where  $\mathbf{y}$  denotes the two dimensional spatial vector ( $y, z$ ) with  $\mathbf{k}$  the corresponding two dimensional wave vector, and  $N$  is a normalization factor to be determined later.  $f(x)$  must satisfy the separated dynamical equation

$$\left[ \frac{d^2}{dx^2} + (\omega - e\Phi)^2 - (\mathbf{k}^2 + m^2) \right] f(x) = 0. \tag{3}$$

Far from the barrier the solutions of the dynamical equation look like plane waves. One complete set of modes is given asymptotically by

$$\vec{w}_{in} = \vec{N}e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \begin{cases} \vec{e}^{iqx} + \vec{R}e^{-iqx} & \text{as } x \rightarrow -\infty \\ \vec{T}e^{irx} & \text{as } x \rightarrow \infty \end{cases} \tag{4a}$$

and

$$\vec{w}_{in} = \vec{N}e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \begin{cases} \vec{T}e^{-iqx} & \text{as } x \rightarrow -\infty \\ e^{-irx} + \vec{R}e^{irx} & \text{as } x \rightarrow \infty, \end{cases} \tag{4b}$$

where

$$\begin{aligned} q^2 &= (\omega + eV/2)^2 - (\mathbf{k}^2 + m^2) \\ r^2 &= (\omega - eV/2)^2 - (\mathbf{k}^2 + m^2). \end{aligned} \tag{5}$$

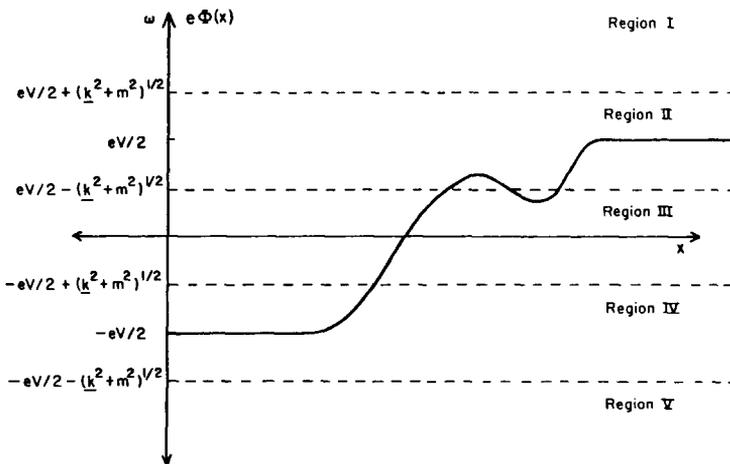


FIG. 1. An example of the type of potential  $\Phi(x)$  considered, showing the five regions for the frequency.

The reflection and transmission coefficients  $\vec{R}$ ,  $\bar{R}$ ,  $\vec{T}$ , and  $\bar{T}$  depend on the detailed shape of the barrier. Some specific examples are calculated in Appendix 1.

The range of frequencies is divided into five regions according to the behavior of waves with those frequencies (see Fig. 1),

$$\begin{array}{ll}
 \text{Region I} & eV/2 + (\mathbf{k}^2 + m^2)^{1/2} \leq \omega \\
 \text{Region II} & eV/2 - (\mathbf{k}^2 + m^2)^{1/2} \leq \omega \leq eV/2 + (\mathbf{k}^2 + m^2)^{1/2} \\
 \text{Region III} & -eV/2 + (\mathbf{k}^2 + m^2)^{1/2} \leq \omega \leq eV/2 - (\mathbf{k}^2 + m^2)^{1/2} \\
 \text{Region IV} & -eV/2 - (\mathbf{k}^2 + m^2)^{1/2} \leq \omega \leq -eV/2 + (\mathbf{k}^2 + m^2)^{1/2} \\
 \text{Region V} & \omega \leq -eV/2 - (\mathbf{k}^2 + m^2)^{1/2}.
 \end{array}$$

Note that  $q$  is imaginary in region IV and  $r$  is imaginary in region II. Consider the first set of modes Eq. (4a), henceforth called left modes. Build a wave packet from left modes with a narrow frequency range. This broad packet consists of a wave which comes in from the left at early times, with at late times both a reflected piece back toward the left and a transmitted piece toward the right. See Fig. 2a. Such a wave can exist only for frequencies  $\omega$  for which  $q$  is real (regions I, II, III, V). When  $q$  is imaginary (region IV) the incoming wave is physically unable to propagate, corresponding to the mass gap in a spacetime with zero potential. When  $r$  is imaginary (region II), the wave exists since the incoming and reflected parts propagate freely; however, the transmitted part is exponentially damped. The second set of modes, Eq. (4b), henceforth called right modes, is classified similarly with the roles of  $q$  and  $r$  reversed; i.e., no modes exist when  $r$  is imaginary (region II) since the incoming wave cannot propagate; when  $q$  is imaginary (region IV), the mode exists but its transmitted part is exponentially damped. See Fig. 2b.

The dispersion relations Eq. (5) do not determine the signs of  $q$  and  $r$ . The group velocity of the waves must be used to determine these signs. We require the

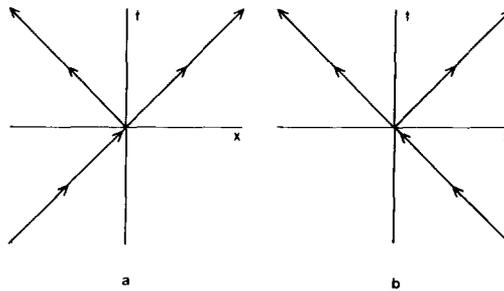


FIG. 2. (a) Left in-modes come in from infinity from the left at early times and go out to infinity in both directions at late times. (b) Right in-modes come in from infinity from the right at early times and go out to infinity in both directions at late times.

incoming part of the wave from the left modes to have positive group velocity  $d\omega/dq$  so that they travel from left to right. The restriction

$$\frac{d\omega}{dq} = \frac{q}{(\omega + eV/2)} \geq 0$$

gives

$$q > 0 \quad \text{for } \omega > -\frac{eV}{2} \quad \text{regions I, II, III}$$

$$q < 0 \quad \text{for } \omega < -\frac{eV}{2} \quad \text{region V.}$$

Similarly requiring the incoming part of the wave from the right modes to travel from right to left gives

$$\frac{d\omega}{dr} = \frac{r}{(\omega - eV/2)} \geq 0$$

$$r > 0 \quad \text{for } \omega > \frac{eV}{2} \quad \text{region I}$$

$$r < 0 \quad \text{for } \omega < \frac{eV}{2} \quad \text{regions III, IV, V.}$$

If  $w$  and  $w'$  are two solutions of the dynamical equation (2), their inner product  $(w, w')$  is defined by

$$(w, w') = -i \int_{\Sigma} (w^* w'^{,\mu} - w^{*\mu} w') d\Sigma_{\mu},$$

where  $\Sigma_{\mu}$  is an arbitrary, asymptotically spacelike hypersurface. The inner product is independent of  $\Sigma_{\mu}$  by virtue of the dynamical equation (2). For simplicity we choose a constant time hypersurface. To normalize the modes it is sufficient to normalize the incoming part of the wave in the asymptotic potential which it sees. This corresponds to taking the plane wave limit of the normalization of a broad packet [3]. The choice

$$\vec{N} = [2(2\pi)^3 |q|]^{-1/2}$$

$$\vec{N} = [2(2\pi)^3 |r|]^{-1/2}$$

normalizes the modes so that

$$\begin{aligned} (\vec{w}_{in}, \vec{w}'_{in}) &= \varepsilon \delta(\omega - \omega') \delta^2(\mathbf{k} - \mathbf{k}') \\ (\vec{w}_{in}, \vec{w}'_{in}) &= \varepsilon' \delta(\omega - \omega') \delta^2(\mathbf{k} - \mathbf{k}') \\ (\vec{w}_{in}, \vec{w}'_{in}) &= 0, \end{aligned} \tag{6}$$

where

$$\varepsilon = \begin{cases} +1 & \text{when } \omega + eV/2 > 0 \\ -1 & \text{when } \omega + eV/2 < 0 \end{cases} \quad (7)$$

$$\varepsilon' = \begin{cases} +1 & \text{when } \omega - eV/2 > 0 \\ -1 & \text{when } \omega - eV/2 < 0. \end{cases}$$

Although the reflection and transmission coefficients depend upon the detailed shape of the barrier, general relations among the coefficients (known as Wronskian relations) can be obtained from current conservation. If  $f_1$  and  $f_2$  are any two solutions of the separated dynamical equation (3), then

$$f_1 \left( \frac{d}{dx} f_2 \right) - \left( \frac{d}{dx} f_1 \right) f_2 \quad (8)$$

is a conserved current because the first derivative of this expression vanishes by virtue of Eq. (3). Note that since Eq. (3) is real, if  $f$  is a solution, so is  $f^*$ . Equating the value of expression (8) in the two asymptotic regions for various solutions gives the Wronskian relations listed as Eqs. (1)–(4), (13), and (17) in Table I. The first two of these relations show that in region III (henceforth called the Klein region) where  $q$  is positive and  $r$  is negative we have

$$|R|^2 \geq 1;$$

i.e., the reflected current is greater than the incident current; the modes in the Klein region are superradiant.

### Fermions

The action for a massive spin  $\frac{1}{2}$  field  $\Psi$  minimally coupled to an electromagnetic potential  $A_\mu$  is

$$S = -i \int d^4x (\Psi^\dagger \gamma^0 \gamma^\mu \Psi_{,\mu} + m \Psi^\dagger \gamma^0 \Psi), \quad (9)$$

where the four  $\gamma^\mu$  are  $4 \times 4$  Dirac matrices satisfying the anticommutation relation  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ . Variation with respect to  $\Psi^\dagger$  leads to the dynamical equation

$$\gamma^\mu \Psi_{,\mu} + m \Psi = 0. \quad (10)$$

One complete set of solutions is given asymptotically by

$$\vec{\psi}_{in} = \vec{N}_i e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \times \begin{cases} e^{iqx} \vec{\chi}_i + e^{-iqx} (\vec{R}_i \vec{\chi}_{1R} + \vec{R}'_i \vec{\chi}_{2R}) & \text{as } x \rightarrow -\infty \\ e^{irx} (\vec{T}_i \vec{\chi}_{1T} + \vec{T}'_i \vec{\chi}_{2T}) & \text{as } x \rightarrow \infty \end{cases} \quad (11a)$$

TABLE I  
Boson Wronskian Relations

Regions I, III, V	Region II
(1) $1 -  \bar{R} ^2 = \frac{r}{q}  \bar{T} ^2$	(13) $ \bar{R} ^2 = 1$
(2) $1 -  \bar{R} ^2 = \frac{q}{r}  \bar{T} ^2$	(14) $ \bar{Q} ^2 = 1$
(3) $\bar{T} = \frac{r}{q} \bar{T}$	(15) $\bar{R}\bar{Q} = 1$
(4) $\bar{R}\bar{T}^* = -\frac{q}{r} \bar{R}^* \bar{T}$	(16) $\bar{R} = \bar{Q}^*$
(5) $1 -  \bar{Q} ^2 = \frac{q}{r}  \bar{S} ^2$	
(6) $1 -  \bar{Q} ^2 = \frac{r}{q}  \bar{S} ^2$	
(7) $\bar{S} = \frac{q}{r} \bar{S}$	
(8) $\bar{Q}\bar{S}^* = -\frac{r}{q} \bar{S}\bar{Q}^*$	
(9) $\bar{R} = \bar{Q}^*$	Region IV
(10) $\bar{R} = \bar{Q}^*$	(17) $ \bar{R} ^2 = 1$
(11) $\bar{T} = \frac{q}{r} \bar{S}^*$	(18) $ \bar{Q} ^2 = 1$
(12) $\bar{T} = \frac{r}{q} \bar{S}^*$	(19) $\bar{R}\bar{Q} = 1$
	(20) $\bar{R} = \bar{Q}^*$

and

$$\begin{aligned} \bar{\psi}_{in} &= \bar{N}_i e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \\ &\times \begin{cases} e^{-iqx} (\bar{T}_i \bar{\chi}_{1T} + \bar{T}'_i \bar{\chi}_{2T}) & \text{as } x \rightarrow -\infty \\ e^{-irx} \bar{\chi}_i + e^{irx} (\bar{R}_i \bar{\chi}_{1R} + \bar{R}'_i \bar{\chi}_{2R}) & \text{as } x \rightarrow \infty, \end{cases} \end{aligned} \tag{11b}$$

where  $i = 1, 2$  is a spin index, the various  $R$ 's and  $T$ 's are reflection and transmission coefficients which depend on the detailed shape of the potential barrier (see Appendix 1), and the  $\chi$ 's are constant spinors. The notation used for the spinors and a description of their properties is given in Appendix 2.

The analysis of the behavior of the modes in the different regions is identical to that of the boson case since the solutions of the fermion dynamical equation (10) are also solutions of the equation

$$\psi_{,\mu}{}^{\mu} - m^2 \psi - ie\gamma^0 \gamma^1 \left( \frac{d}{dx} \Phi \right) \psi = 0.$$

This is just the dynamical equation for the boson case with an extra term containing the derivative of the barrier function. Since the barrier function is constant in the asymptotic regions this term does not affect the dispersion relations Eq. (5) or the analysis of the group velocity of the waves.

If  $\psi$  and  $\psi'$  are two solutions of the dynamical equation, their inner product  $(\psi, \psi')$  is defined by

$$(\psi, \psi') = -i \int_{\Sigma} (\psi^\dagger \gamma^0 \gamma^\mu \psi') d\Sigma_\mu,$$

where  $\Sigma_\mu$  is an arbitrary spacelike hypersurface, chosen for simplicity to be a constant time hypersurface. The inner product is independent of the choice of hypersurface by virtue of the dynamical equation (10). As before we will normalize the incoming part of the wave in the asymptotic potential which it sees. The choice

$$\begin{aligned} \vec{N}_i &= (2\pi)^{-3/2} \left( \frac{m}{|q|} \right)^{1/2} \\ \bar{N}_i &= (2\pi)^{-3/2} \left( \frac{m}{|r|} \right)^{1/2} \end{aligned}$$

normalizes the modes so that

$$(\psi_i, \psi'_j) = +\delta(\omega - \omega') \delta^2(\mathbf{k} - \mathbf{k}') \delta_{ij} \delta_{\leftarrow},$$

where  $\delta_{\leftarrow}$  means that the inner product is zero unless the modes are both incoming from the left or both incoming from the right. Note that the plus sign holds for all modes, unlike the boson case.

Wronskian relations for the fermion case can be found from the current  $\psi_i^\dagger \gamma^0 \gamma^1 \psi_j$  which is conserved by virtue of the dynamical equation (10). Using the same methods as those in the boson case and the properties of the spinors discussed in Appendix 2, we obtain the Wronskian relations (valid for  $i, j = 1, 2$ ) listed as Eqs. (1)–(4), (13), and (17) in Table II. Since  $\varepsilon q = |q|$  and  $\varepsilon' r = |r|$ , the first two of these relations with  $i = j$  show that the reflected current is always less than the incident current. Superradiance does not occur in the fermion case.

At this stage it is necessary to discuss previous work on the Klein paradox. Klein's original paper [4] (in German) was the first to consider the solution of the Dirac equation in the presence of a step potential (a particular example of the type of potential considered here). Klein was surprised that there were modes for which the "norm" of the incoming part was positive and the "norm" of the transmitted part was negative. This phenomenon was soon dubbed the Klein paradox by Sauter [5]. Niels Bohr had apparently pointed out to Klein (see the footnote in Klein's paper) the argument using group velocities to determine the signs of  $q$  and  $r$ . Therefore Klein did his calculations correctly and his equations show no superradiance for this fermion case. Never having heard of superradiance and thus

TABLE II  
Fermion Wronskian Relations

Regions I, III, V	Regions I, III, V
(1) $1 - \bar{R}_i^* \bar{R}_j - \bar{R}_i^* \bar{R}_j' = \frac{\epsilon' r}{\epsilon q} (\bar{T}_i^* \bar{T}_j + \bar{T}_i'^* \bar{T}_j')$	(11) $\bar{T}_1 = \frac{\epsilon q}{\epsilon' r} \bar{S}_1^*$
(2) $1 - \bar{R}_i^* \bar{R}_j - \bar{R}_i^* \bar{R}_j' = \frac{\epsilon q}{\epsilon' r} (\bar{T}_i^* \bar{T}_j + \bar{T}_i'^* \bar{T}_j')$	$\bar{T}_1' = \frac{\epsilon q}{\epsilon' r} \bar{S}_2^*$
(3) $\bar{T}_1 = \frac{\epsilon' r}{\epsilon q} \bar{T}_2'$	$\bar{T}_2 = \frac{\epsilon q}{\epsilon' r} \bar{S}_1'^*$
$\bar{T}_1' = \frac{\epsilon' r}{\epsilon q} \bar{T}_1$	$\bar{T}_2' = \frac{\epsilon q}{\epsilon' r} \bar{S}_2'^*$
$\bar{T}_2 = \frac{\epsilon' r}{\epsilon q} \bar{T}_2'$	(12) $\bar{T}_1 = \frac{\epsilon' r}{\epsilon q} \bar{S}_1^*$
$\bar{T}_2' = \frac{\epsilon' r}{\epsilon q} \bar{T}_1$	$\bar{T}_1' = \frac{\epsilon' r}{\epsilon q} \bar{S}_2^*$
(4) $\bar{R}_i \bar{T}_j^* + \bar{R}_i' \bar{T}_j'^* = -\frac{\epsilon q}{\epsilon' r} (\bar{R}_j^* \bar{T}_i + \bar{R}_j'^* \bar{T}_i')$	$\bar{T}_2 = \frac{\epsilon' r}{\epsilon q} \bar{S}_1'^*$
(5) $1 - \bar{Q}_i^* \bar{Q}_j - \bar{Q}_i'^* \bar{Q}_j' = \frac{\epsilon q}{\epsilon' r} (\bar{S}_i^* \bar{S}_j + \bar{S}_i'^* \bar{S}_j')$	$\bar{T}_2' = \frac{\epsilon' r}{\epsilon q} \bar{S}_2'^*$
(6) $1 - \bar{Q}_i^* \bar{Q}_j - \bar{Q}_i'^* \bar{Q}_j' = \frac{\epsilon' r}{\epsilon q} (\bar{S}_i^* \bar{S}_j + \bar{S}_i'^* \bar{S}_j')$	
(7) $\bar{S}_1 = \frac{\epsilon q}{\epsilon' r} \bar{S}_2'$	<b>Region II</b>
$\bar{S}_1' = \frac{\epsilon q}{\epsilon' r} \bar{S}_1$	(13) $\bar{R}_i^* \bar{R}_j + \bar{R}_i'^* \bar{R}_j' = 1$
$\bar{S}_2 = \frac{\epsilon q}{\epsilon' r} \bar{S}_2$	(14) $\bar{Q}_i^* \bar{Q}_j + \bar{Q}_i'^* \bar{Q}_j' = 1$
$\bar{S}_2' = \frac{\epsilon q}{\epsilon' r} \bar{S}_1$	(15) $(1 - \delta_{ij}) = \bar{R}_i \bar{Q}_j' + \bar{R}_i' \bar{Q}_j$
(8) $\bar{Q}_i \bar{S}_j^* + \bar{Q}_i' \bar{S}_j'^* = -\frac{\epsilon' r}{\epsilon q} (\bar{Q}_j^* \bar{S}_i + \bar{Q}_j'^* \bar{S}_i')$	(16) $\bar{Q}_1 = \bar{R}_1^*$
(9) $\bar{R}_1 = \bar{Q}_1^*$	$\bar{Q}_1' = \bar{R}_2^*$
$\bar{R}_1' = \bar{Q}_2^*$	$\bar{Q}_2 = \bar{R}_1^*$
$\bar{R}_2 = \bar{Q}_1'^*$	$\bar{Q}_2' = \bar{R}_2^*$
$\bar{R}_2' = \bar{Q}_2'^*$	
(10) $\bar{R}_1 = \bar{Q}_1^*$	<b>Region IV</b>
$\bar{R}_1' = \bar{Q}_2^*$	(17) $\bar{R}_i^* \bar{R}_j + \bar{R}_i'^* \bar{R}_j' = 1$
$\bar{R}_2 = \bar{Q}_1'^*$	(18) $\bar{Q}_i^* \bar{Q}_j + \bar{Q}_i'^* \bar{Q}_j' = 1$
$\bar{R}_2' = \bar{Q}_2'^*$	(19) $(1 - \delta_{ij}) = \bar{R}_i \bar{Q}_j' + \bar{R}_i' \bar{Q}_j$
	(20) $\bar{Q}_1 = \bar{R}_1^*$
	$\bar{Q}_1' = \bar{R}_2^*$
	$\bar{Q}_2 = \bar{R}_1^*$
	$\bar{Q}_2' = \bar{R}_2^*$

not expecting it, he made no comment about this point. Much later, workers publishing in English, including the standard textbook treatments [6], missed the argument about the signs of  $q$  and  $r$  and assumed them to be positive. As the reader can see, this would make the fermion modes in the Klein region appear superradiant. The name Klein paradox was reassigned to this mistaken phenomenon of superradiance. Interestingly, there are superradiant boson modes, so the effect does exist but for a different example. The most recent papers dealing with the Klein paradox are of several types. They consider only the boson case [7]; or they contain the same mistaken sign in various forms [8]; or they obtain the correct sign but misinterpret the significance of the norm of the modes [9]; or they put the correct sign into the Wronskian relations without proof [10] so as to agree with the Kerr black hole calculations which show superradiant boson modes, but no superradiant fermion modes. After this work was completed, a nice treatment of superradiance was published by Greiner, Müller, and Rafelski [14] which gets both the signs and the physical argument correct.

### III. SECOND QUANTIZATION, CURRENT, MOMENTUM

Further insight into the Klein paradox is obtained when the problem is second quantized and viewed in the many particle picture.

#### *Bosons*

During canonical quantization the field  $\phi$  is usually written as a sum over the positive frequency solutions of the dynamical field equation and over the complex conjugate of the positive frequency solutions to the complex conjugate of the dynamical field equation. However, when an external potential is present modes with positive and negative frequency do not necessarily correspond to modes with positive and negative norm. It is actually the norm of the mode which is important as it is this sign which determines the commutation relations between the creation and the annihilation operators. To avoid confusion this paper will always expand the field as a sum over *all* the solutions of the dynamical field equation (which is exactly equivalent to the standard method). However, the reader is cautioned to remember that integrals over the frequency  $\omega$  now extend over all values from minus infinity to infinity which are allowed by the dispersion relations Eqs. (5) rather than over the allowed values from zero to infinity as in the standard procedure. The integral over  $\mathbf{k}$  also extends from minus infinity to infinity in both dimensions unless otherwise specified,

$$\phi(x) = \sum_{\leftrightarrow} \int d\omega \int d^2\mathbf{k} [a_{\text{in}}(\omega, \mathbf{k}, \leftrightarrow) u_{\text{in}}(\omega, \mathbf{k}, \leftrightarrow | x) + b_{\text{in}}^\dagger(\omega, \mathbf{k}, \leftrightarrow) v_{\text{in}}(\omega, \mathbf{k}, \leftrightarrow | x)], \quad (12)$$

where the  $u_{in}$  and  $v_{in}$  are the positive and negative normed modes  $w_{in}$ , respectively, and  $\leftrightarrow$  is a label which runs over left and right modes.

Now according to the canonical procedure,  $\phi$  and

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_0} = -\dot{\phi}^* \cdot 0$$

become operators which satisfy the commutation relation

$$[\phi(t, x, \mathbf{y}), \pi(t, x', \mathbf{y}')] = i\delta(x - x') \delta^2(\mathbf{y} - \mathbf{y}')$$

This imposes the following commutation relations on the creation operators  $a_{in}$  and  $b_{in}$ ,

$$\begin{aligned} [a_{in}(\omega, \mathbf{k}, \leftrightarrow), a_{in}^\dagger(\omega', \mathbf{k}', \leftrightarrow)] &= \delta(\omega - \omega') \delta^2(\mathbf{k} - \mathbf{k}') \delta_{\leftrightarrow}, \\ [b_{in}(\omega, \mathbf{k}, \leftrightarrow), b_{in}^\dagger(\omega', \mathbf{k}', \leftrightarrow)] &= \delta(\omega - \omega') \delta^2(\mathbf{k} - \mathbf{k}') \delta_{\leftrightarrow}. \end{aligned}$$

All other commutators are zero.

The charge symmetric form of the current operator (obtained by varying the action Eq. (1) with respect to  $A_\mu$ ) is given by

$$\begin{aligned} j^\mu &= \left(\frac{-ie}{2}\right) \lim_{x \rightarrow x'} (\partial^\mu - \partial^{\mu'} - ieA^\mu - ieA^{\mu'}) \\ &\times [\phi(t, x, \mathbf{y}) \phi^*(t', x', \mathbf{y}') + \phi^*(t', x', \mathbf{y}') \phi(t, x, \mathbf{y})]. \end{aligned}$$

Using the expansion of  $\phi$ , Eq. (12), and taking the vacuum expectation value, one obtains

$$\begin{aligned} \langle 0_{in} | j^\mu | 0_{in} \rangle &= \left(\frac{-ie}{2}\right) \lim_{x \rightarrow x'} (\partial^\mu - \partial^{\mu'} - ieA^\mu - ieA^{\mu'}) \\ &\times \sum_{\leftrightarrow} \int d\omega \int d^2\mathbf{k} [u_{in}(\omega, \mathbf{k}, \leftrightarrow | x) u_{in}^*(\omega, \mathbf{k}, \leftrightarrow | x') \\ &+ v_{in}(\omega, \mathbf{k}, \leftrightarrow | x) v_{in}^*(\omega, \mathbf{k}, \leftrightarrow | x')], \end{aligned}$$

where the in-vacuum  $|0_{in}\rangle$  is defined by

$$\begin{aligned} a_{in}(\omega, \mathbf{k}, \leftrightarrow) |0_{in}\rangle &= 0 \\ b_{in}(\omega, \mathbf{k}, \leftrightarrow) |0_{in}\rangle &= 0 \end{aligned}$$

After inserting the expressions for the modes Eqs. (4) into the above expression, a straightforward calculation using the Wronskian relations in Table I gives for the current in the  $x$ -direction

$$\langle 0_{in} | j^x | 0_{in} \rangle = \frac{e}{(2\pi)^3} \int d^2\mathbf{k} \int d\omega \frac{r}{q} |\bar{T}|^2 \quad \text{as } x \rightarrow \pm\infty, \quad (13)$$

where the integrals here are over only those modes in the Klein region. In the other regions the contribution from the left modes cancels the contribution from the right modes. Since  $r/q$  is negative in region III we see there is a current traveling from right to left indicating a flow of charged particles. To examine which way the particles are traveling we will calculate the vacuum expectation value of the momentum  $T^{ix}$ . The stress-energy tensor  $T^{\mu\nu}$  (obtained by varying the action Eq. (1) with respect to  $g_{\mu\nu}$  is

$$T^{\mu\nu} = -g^{\mu\nu}[\phi^{*\cdot\rho}\phi_{\cdot\rho} + m^2\phi^*\phi] + \phi^{*\cdot\mu}\phi_{\cdot\nu} + \phi^{*\cdot\nu}\phi_{\cdot\mu}.$$

With a calculation similar to that for the current we obtain

$$\begin{aligned} \langle 0_{\text{in}} | T^{ix} | 0_{\text{in}} \rangle &= -\frac{1}{2} \lim_{x \rightarrow x'} [(\partial_t - ie\Phi)(\partial_x) + (\partial_t + ie\Phi)(\partial_{x'})] \\ &\quad \times \sum_{\leftrightarrow} \int d\omega \int d^2\mathbf{k} [u_{\text{in}}(\omega, \mathbf{k}, \leftrightarrow | x) u_{\text{in}}^*(\omega, \mathbf{k}, \leftrightarrow | x') \\ &\quad + v_{\text{in}}(\omega, \mathbf{k}, \leftrightarrow | x) v_{\text{in}}^*(\omega, \mathbf{k}, \leftrightarrow | x')] \\ &= \begin{cases} +\frac{1}{(2\pi)^3} \int d^2\mathbf{k} \int d\omega \left(\omega + \frac{eV}{2}\right) \frac{r}{q} |\vec{T}|^2 & \text{as } x \rightarrow -\infty \\ +\frac{1}{(2\pi)^3} \int d^2\mathbf{k} \int d\omega \left(\omega - \frac{eV}{2}\right) \frac{r}{q} |\vec{T}|^2 & \text{as } x \rightarrow \infty, \end{cases} \end{aligned}$$

where again the integrals are over only those modes in the Klein region. These equations show that the momentum is negative to the left of the barrier and positive to the right of the barrier. Thus we see that charged pairs are being created at the potential barrier. The positively charged particles are being repelled off toward the left and the negatively charged particles are repelled off toward the right, creating a net negative current everywhere.

### Fermions

Expand the field  $\Psi$  as a sum over the complete set of solutions of the dynamical equation

$$\begin{aligned} \Psi(x) &= \sum_{\leftrightarrow} \sum_{i=1,2} \int d\omega \int d^2\mathbf{k} [c_{\text{in}}(\omega, \mathbf{k}, \leftrightarrow, i) u_{i\text{in}}(\omega, \mathbf{k}, \leftrightarrow | x) \\ &\quad + d_{\text{in}}^\dagger(\omega, \mathbf{k}, \leftrightarrow, i) v_{i\text{in}}(\omega, \mathbf{k}, \leftrightarrow | x)], \end{aligned} \quad (14)$$

where the division between the  $u_{i\text{in}}$  and  $v_{i\text{in}}$  occurs for precisely the same frequencies and directions of modes as those in the boson case.

Now according to the canonical procedure  $\Psi$  and

$$\Pi \equiv \frac{\partial \mathcal{L}}{\partial \Psi_0} = -i\Psi^\dagger$$

become operators which satisfy the anticommutation relations

$$\{\Psi(t, x, \mathbf{y}), \Pi(t, x', \mathbf{y}')\} = i\delta(x - x') \delta^2(\mathbf{y} - \mathbf{y}')$$

which imposes the following anticommutation relations on the annihilation operators  $c_{in}$  and  $d_{in}$ ,

$$\begin{aligned} \{c_{in}(\omega, \mathbf{k}, \leftrightarrow, i), c_{in}^\dagger(\omega', \mathbf{k}', \leftrightarrow, j)\} &= \delta(\omega - \omega') \delta^2(\mathbf{k} - \mathbf{k}') \delta_{\leftrightarrow} \delta_{ij} \\ \{d_{in}(\omega, \mathbf{k}, \leftrightarrow, i), d_{in}^\dagger(\omega', \mathbf{k}', \leftrightarrow, j)\} &= \delta(\omega - \omega') \delta^2(\mathbf{k} - \mathbf{k}') \delta_{\leftrightarrow} \delta_{ij}. \end{aligned}$$

All other anticommutators are zero.

The current operator (obtained by varying the action Eq. (9) with respect to  $A_\mu$ ) is given by

$$j^\mu = -e\Psi^\dagger\gamma^0\gamma^\mu\Psi.$$

Charge symmetrizing, using the expansion for  $\Psi$  Eq. (14), and taking the vacuum expectation value, one obtains

$$\begin{aligned} \langle 0_{in} | j^x | 0_{in} \rangle &= -\frac{e}{2} \sum_{\leftrightarrow} \sum_i \int d\omega \int d^2\mathbf{k} \\ &\times [v_{in}^\dagger(\omega, \mathbf{k}, \leftrightarrow | x) \gamma^0 \gamma^1 v_{in}(\omega, \mathbf{k}, \leftrightarrow | x) \\ &- u_{in}^\dagger(\omega, \mathbf{k}, \leftrightarrow | x) \gamma^0 \gamma^1 u_{in}(\omega, \mathbf{k}, \leftrightarrow | x)]. \end{aligned} \quad (15)$$

After inserting the expressions for the modes Eqs. (11) into the above expression, a straightforward calculation using the Wronskian relations in Table II and the properties of the spinors and projection operators discussed in Appendix 1 gives for the current in the  $x$ -direction

$$\begin{aligned} \langle 0_{in} | j^x | 0_{in} \rangle &= -\frac{e}{(2\pi)^3} \int d^2\mathbf{k} \int d\omega \frac{\varepsilon' r}{\varepsilon q} \\ &\times (|\bar{T}_1|^2 + |\bar{T}'_1|^2 + |\bar{T}_2|^2 + |\bar{T}'_2|^2) \quad \text{as } x \rightarrow \pm\infty, \end{aligned} \quad (16)$$

where the integrals are over only those modes in the Klein region. Since  $\varepsilon' r/\varepsilon q$  is always positive we see there is a current traveling from right to left, cf. Eq. (13).

The stress-energy tensor  $T^{\mu\nu}$  (obtained by varying the action Eq. (9) with respect to the tetrad  $e_{a\mu}$  [11]) is

$$T^{\mu\nu} = \frac{i}{4} (\Psi^\dagger \gamma^0 \gamma^\mu \Psi^{,\nu} + \Psi^\dagger \gamma^0 \gamma^\nu \Psi^{,\mu} - \Psi^{\dagger,\nu} \gamma^0 \gamma^\mu \Psi - \Psi^{\dagger,\mu} \gamma^0 \gamma^\nu \Psi).$$

With a calculation similar to that for the current we obtain

$$\begin{aligned} \langle 0_{in} | T^{ix} | 0_{in} \rangle &= -\frac{1}{(2\pi)^3} \int d^2\mathbf{k} d\omega \left( \omega + \frac{eV}{2} \right) \frac{\varepsilon' r}{\varepsilon q} \\ &\times (|\bar{T}_1|^2 + |\bar{T}'_1|^2 + |\bar{T}_2|^2 + |\bar{T}'_2|^2) \quad \text{as } x \rightarrow -\infty \end{aligned}$$

$$\langle 0_{in} | T^{ix} | 0_{in} \rangle = -\frac{1}{(2\pi)^3} \int d^2\mathbf{k} \int d\omega \left( \omega - \frac{eV}{2} \right) \frac{\varepsilon' r}{\varepsilon q} \times (|\vec{T}_1|^2 + |\vec{T}'_1|^2 + |\vec{T}_2|^2 + |\vec{T}'_2|^2) \quad \text{as } x \rightarrow +\infty,$$

where the integrals are over only those modes in the Klein region. As in the boson case, the momentum to the left is negative and the momentum to the right is positive indicating charged particles created at the barrier being repelled away in opposite directions creating a negative current everywhere.

IV. BOGOLIUBOV RELATIONS, NUMBER OF PARTICLES

*Bosons*

A second complete set of solutions to the dynamical equation (2) is given asymptotically by

$$\vec{w}_{out} = \vec{M} e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \times \begin{cases} e^{-iqx} + \tilde{Q} e^{iqx} & \text{as } x \rightarrow -\infty \\ \tilde{S} e^{-irx} & \text{as } x \rightarrow \infty \end{cases} \quad (17a)$$

and

$$\vec{w}_{out} = \vec{M} e^{i(\mathbf{k} \cdot \mathbf{y} - \omega t)} \times \begin{cases} \tilde{S} e^{iqx} & \text{as } x \rightarrow -\infty \\ e^{irx} + \tilde{Q} e^{-irx} & \text{as } x \rightarrow \infty. \end{cases} \quad (17b)$$

The analysis of the behavior and properties of these modes is analogous to that given for the in-modes in Section II. A wave packet built from a narrow frequency range of the modes in Eq. (17a) will travel in toward the barrier from both sides at early times and scatter off only toward the left at late times (see Fig. 3a). At late times a wave packet built from the modes in Eq. (17b) will scatter off only toward the right (see Fig. 3b).

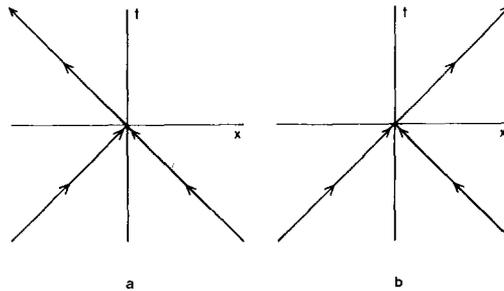


FIG. 3. (a) Left out-modes come in from infinity from both sides at early times and go out to infinity to the left at late times. (b) Right out-modes come in from infinity from both sides at early times and go out to infinity to the right at late times.

To normalize the modes, normalize the outgoing part of the wave in the asymptotic potential which it sees. The choice

$$\begin{aligned} \vec{M} &= [2(2\pi)^3 |q|]^{-1/2} \\ \vec{M} &= [2(2\pi)^3 |r|]^{-1/2} \end{aligned}$$

normalizes the modes so that

$$\begin{aligned} (\vec{w}_{out}, \vec{w}'_{out}) &= \epsilon' \delta(\omega - \omega') \delta^2(\mathbf{k} - \mathbf{k}') \\ (\vec{w}_{out}, \vec{w}'_{out}) &= \epsilon \delta(\omega - \omega') \delta^2(\mathbf{k} - \mathbf{k}') \\ (\vec{w}_{out}, \vec{w}'_{out}) &= 0. \end{aligned} \tag{18}$$

Equations (5)–(12), (14)–(16), and (18)–(20) in Table I give Wronskian relations for the out-modes, and between in- and out-modes.

Since the out-modes and in-modes separately form complete sets of solutions of the dynamical equation, we can write one set as a linear combination of the other, i.e.,

$$\begin{aligned} u_{iout} &= \sum_j \alpha_{ij} u_{j\text{in}} + \sum_q \beta_{iq} v_{q\text{in}} \\ v_{pout} &= \sum_j \gamma_{pj} u_{j\text{in}} + \sum_q \epsilon_{pq} v_{q\text{in}}, \end{aligned} \tag{19}$$

where the  $\alpha_{ij}$ ,  $\beta_{iq}$ ,  $\gamma_{pj}$ , and  $\epsilon_{pq}$  are known as Bogoliubov coefficients [12] and the indicies  $i, j, p$ , and  $q$  stand for all the labels  $w, \mathbf{k}, \leftrightarrow$ , and the sum becomes an integral over continuous indices. If we equate the two expansions of  $\phi$  in terms of the in-modes Eqs. (4) and the out-modes Eqs. (17) and use the orthonormality properties of the modes Eqs. (6) and (18), we can express the out-annihilation and creation operators in terms of the in-annihilation and creation operators,

$$\begin{aligned} a_{iout} &= \sum_j \alpha_{ij}^* a_{j\text{in}} - \sum_q \beta_{iq}^* b_{q\text{in}} \\ b_{pout} &= -\sum_j \gamma_{pj} a_{j\text{in}} + \sum_q \epsilon_{pq} b_{q\text{in}}. \end{aligned} \tag{20}$$

We can then ask how many out-type particles exist in the in-vacuum (i.e., how many particles exist at late times in a state which has no particles at early times). Using Eq. (20) we obtain

$$\begin{aligned} \langle 0_{in} | a_{iout}^\dagger a_{iout} | 0_{in} \rangle &= \sum_q \beta_{iq} \beta_{qi}^* \\ \langle 0_{in} | b_{pout}^\dagger b_{pout} | 0_{in} \rangle &= \sum_j \gamma_{pj}^* \gamma_{jp}. \end{aligned} \tag{21}$$

We see that unless  $\beta_{ip}$  or  $\gamma_{pi}$  are nonzero the in-vacuum and the out-vacuum are equivalent (i.e., no particles are produced).

For the types of potentials we are considering, it is clear that each out-mode is a linear combination of just the one or two in-modes with the same frequency  $\omega$  and wave vector  $\mathbf{k}$ . In regions I, II, IV, and V the out-modes have the same norm as the in-modes of which they are a linear combination, and thus  $\beta_{ip}$  and  $\gamma_{pi}$  are zero in these regions. In the Klein region, however, the two in-modes have opposite norm, as do the two out-modes. Therefore  $\beta_{ip}$  and  $\gamma_{pi}$  are nonzero in this region. A simple calculation comparing the modes in the asymptotic regions gives

$$\beta_{ip} \equiv \beta(\omega, \mathbf{k}, \leftarrow | \omega', \mathbf{k}', \leftarrow) = \left| \frac{r}{q} \right|^{1/2} \bar{T}^* \delta(\omega - \omega') \delta^2(\mathbf{k} - \mathbf{k}')$$

$$\gamma_{pi} \equiv \gamma(\omega, \mathbf{k}, \rightarrow | \omega', \mathbf{k}', \rightarrow) = - \left| \frac{r}{q} \right|^{1/2} \bar{T}^* \delta(\omega - \omega') \delta^2(\mathbf{k} - \mathbf{k}'),$$

showing that the total number of out-particles in the in-vacuum is

$$\sum_i \langle 0_{in} | a_{iout}^\dagger a_{iout} | 0_{in} \rangle = \frac{1}{(2\pi)^3} \int d\omega \int d^2\mathbf{k} \left| \frac{r}{q} \right| |\bar{T}|^2 \delta^3(0)$$

$$\sum_p \langle 0_{in} | b_{pout}^\dagger b_{pout} | 0_{in} \rangle = \frac{1}{(2\pi)^3} \int d\omega \int d^2\mathbf{k} \left| \frac{r}{q} \right| |\bar{T}|^2 \delta^3(0),$$
(22)

where the integrals are over the modes in the Klein region.

The infinite  $\delta^3(0)$  factor arises because the potential is constant in the infinite  $y$ - and  $z$ -directions and for an infinite time. If the potential were placed in a box periodic in the  $y$ - and  $z$ -directions and in time, then the  $\omega$  and  $\mathbf{k}$  integrals would be replaced by sums and the  $\delta^3(0)$  factor would be replaced by  $L^2 T / (2\pi)^3$  where  $L^2$  is the area of the box and  $T$  is the periodicity in time. This can be seen most easily by replacing one set of delta functions in the calculation leading to (22) by their Fourier transforms. Dividing by  $L^2 T$  gives an (asymptotic) expression for the number of particles passing through a unit area normal to the  $x$ -direction in a unit time. This expression is just what one would expect based on the results for the current Eq. (13) in Section III.

### Fermions

Fermion Bogoliubov relations are defined analogously to the boson relations. Equation (19) is formally identical but with the indices  $i, j, p$ , and  $q$  also including the spin label. Since all the modes have positive norm, Eq. (20) becomes

$$a_{iout} = \sum_j \alpha_{ij}^* a_{jin} + \sum_q \beta_{iq}^* b_{qin}^{\dagger}$$

$$b_{pout} = \sum_j \gamma_{pj} a_{jin}^{\dagger} + \sum_q \varepsilon_{pq} b_{qin},$$

leading to an expression formally identical to Eq. (21). A simple calculation comparing modes in the asymptotic region gives in the Klein region

$$\begin{aligned} \beta_{ip} &\equiv \beta(\omega, \mathbf{k}, \leftarrow | \omega', \mathbf{k}', \leftarrow) \\ &= \left| \frac{r}{q} \right|^{1/2} (|\vec{T}_1|^2 + |\vec{T}'_1|^2 + |\vec{T}_2|^2 + |\vec{T}'_2|^2) \delta(\omega - \omega') \delta^2(\mathbf{k} - \mathbf{k}') \\ \gamma_{pi} &\equiv \gamma(\omega, \mathbf{k}, \rightarrow | \omega', \mathbf{k}', \rightarrow) \\ &= \left| \frac{r}{q} \right|^{1/2} (|\vec{T}_1|^2 + |\vec{T}'_1|^2 + |\vec{T}_2|^2 + |\vec{T}'_2|^2) \delta(\omega - \omega') \delta^2(\mathbf{k} - \mathbf{k}'), \end{aligned}$$

giving

$$\begin{aligned} \sum_i \langle 0_{in} | a_{iout}^\dagger a_{iout} | 0_{in} \rangle &= \frac{1}{(2\pi)^3} \int d\omega \int d^2\mathbf{k} \left| \frac{r}{q} \right| (|\vec{T}_1|^2 + |\vec{T}'_1|^2 + |\vec{T}_2|^2 + |\vec{T}'_2|^2) \delta^3(\mathbf{0}) \\ \sum_p \langle 0_{in} | b_{pout}^\dagger b_{pout} | 0_{in} \rangle &= \frac{1}{(2\pi)^3} \int d\omega \int d^2\mathbf{k} \left| \frac{r}{q} \right| (|\vec{T}_1|^2 + |\vec{T}'_1|^2 + |\vec{T}_2|^2 + |\vec{T}'_2|^2) \delta^3(\mathbf{0}) \end{aligned}$$

for the total number of out-type particles in the in-vacuum, cf. Eq. (16).

### V. CONCLUSION

In conclusion we see that boson modes are superradiant in the presence of very strong external electromagnetic fields while fermion modes are not. However, the external field is capable of creating pairs of either bosons or fermions. The functional forms, in terms of reflection and transmission coefficients, of the vacuum expectation values of the current, momentum, and number operators agree when used as measures of the number of particles produced.

### APPENDIX 1

All of the expressions for the number of particles created discussed in the body of the paper involve an integral, over modes in the Klein region, of the transmission coefficient for the modes. These transmission coefficients depend on the detailed shape of the barrier. In this appendix we record the transmission coefficients for some specific barriers.

*The Step Potential*  $\Phi(x) = V\theta(x) - V/2$

Using continuity of the wavefunction and its first derivative at  $x=0$  to match the asymptotic values for the modes Eqs. (4), we obtain for the boson case

$$|\tilde{T}|^2 = \frac{4q^2}{(q+r)^2}.$$

Note that when  $\omega=0$ ,  $q=-r$ , making the transmission coefficient infinite, i.e., an infinite particle production rate. This is a property particular to step potentials. The transmission coefficient is determined by the distance the mode must tunnel through the "mass gap." Because of the sharp discontinuity associated with a step potential, the mode does not have to tunnel for any distance at all, allowing for such large transmission coefficients.

Using continuity of the wavefunction, and the known value for the discontinuity of its first derivative at  $x=0$  to match the asymptotic values of the modes Eqs. (11), we obtain in the fermion case

$$|\tilde{T}'_1|^2 + |\tilde{T}'_1|^2 + |\tilde{T}'_2|^2 + |\tilde{T}'_2|^2 = \frac{8q^2\epsilon\epsilon'}{(q+r)^2 - e^2V^2}. \quad (23)$$

The denominator on the right-hand side of Eq. (23) can be written using the dispersion relations Eqs. (5) as  $2\omega^2 - e^2V^2/2 - 2(\mathbf{k}^2 + m^2) + 2qr$ . Since in the Klein region  $q$  is positive,  $r$  is negative, and the maximum value of  $\omega^2$  is  $\omega_{\max}^2 = (eV/2 - (\mathbf{k}^2 + m^2)^{1/2})^2$ , we see that the denominator is strictly negative. In this case the transmission coefficient is never infinite in spite of the sharp discontinuity in the potential. Due to the exclusion principle, whenever a pair of particles has been created further particle creation is repressed.

*The Hyperbolic Tangent Potential*  $\Phi(x) = (V/2) \tanh(lx)$

The solutions of Eq. (3) in the presence of a hyperbolic tangent potential are hypergeometric functions. By comparing the asymptotic forms of the hypergeometric function solutions to the asymptotic forms for the modes in Eqs. (4) we obtain in the boson case

$$|\tilde{T}|^2 = -\frac{q}{r} \frac{\cosh(\pi/l)(q-r) - \cosh(\pi/l)(q+r)}{\cosh \pi(e^2V^2/l^2 - 1)^{1/2} + \cosh(\pi/l)(q+r)}.$$

*The WKB Solution*

An approximate solution to Eq. (3) can be obtained for any sufficiently smooth potential by the WKB method [13]. Comparing these results with the asymptotic solutions for the modes Eqs. (4) we obtain in the boson case

$$|\tilde{T}|^2 = -\frac{q}{r} \operatorname{csch}^2(\ln 2 + \eta),$$

where

$$\eta = \int dx [(\mathbf{k}^2 + m^2) - (\omega - e\Phi(x))^2]^{1/2}$$

integrated over values of  $x$  for which the expression in parentheses is positive (i.e., over the range in which the mode is tunneling through the mass gap).

## APPENDIX 2

To understand the notation used for the spinors it is necessary to define a number of other quantities. First, introduce the operators

$$P_0 = \frac{1}{2}(i\gamma^0 + 1)$$

$$\sigma_0 = -i\gamma^5\gamma^1.$$

Choose four spinors  $\chi_i$ ,  $i = 1, 4$ , such that

$$\chi_i^\dagger \chi_j = \delta_{i,j}$$

$$P_0 \left( \frac{1 + \sigma_0}{2} \right) \chi_1 = \chi_1$$

$$P_0 \left( \frac{1 - \sigma_0}{2} \right) \chi_2 = \chi_2$$

$$(1 - P_0) \left( \frac{1 - \sigma_0}{2} \right) \chi_3 = \chi_3$$

$$(1 - P_0) \left( \frac{1 + \sigma_0}{2} \right) \chi_4 = \chi_4.$$

Such  $\chi$ 's are guaranteed to exist because  $P_0$  and  $\sigma_0$  are hermitian and commute with each other. Now introduce the energy operators

$$\bar{P} = -\frac{1}{2m} \left[ iq\gamma^1 - i\gamma^0 \left( \omega + \frac{eV}{2} \right) + i\mathbf{k} \cdot \boldsymbol{\gamma} - m \right]$$

$$\bar{P}_R = \bar{P}_T = -\frac{1}{2m} \left[ -iq\gamma^1 - i\gamma^0 \left( \omega + \frac{eV}{2} \right) + i\mathbf{k} \cdot \boldsymbol{\gamma} - m \right]$$

$$\bar{P}_T = \bar{P}_R = -\frac{1}{2m} \left[ ir\gamma^1 - i\gamma^0 \left( \omega - \frac{eV}{2} \right) + i\mathbf{k} \cdot \boldsymbol{\gamma} - m \right]$$

$$\bar{P} = -\frac{1}{2m} \left[ -ir\gamma^1 - i\gamma^0 \left( \omega - \frac{eV}{2} \right) + i\mathbf{k} \cdot \boldsymbol{\gamma} - m \right]$$

and the helicity operators

$$\vec{\sigma} = \frac{i}{(\mathbf{k}^2 + m^2)^{1/2}} \left[ q\gamma^5\gamma^0 - \left( \omega + \frac{eV}{2} \right) \gamma^5\gamma^1 \right]$$

$$\vec{\sigma}_R = \vec{\sigma}_T = \frac{i}{(\mathbf{k}^2 + m^2)^{1/2}} \left[ -q\gamma^5\gamma^0 - \left( \omega + \frac{eV}{2} \right) \gamma^5\gamma^1 \right]$$

$$\vec{\sigma}_T = \vec{\sigma}_R = \frac{i}{(\mathbf{k}^2 + m^2)^{1/2}} \left[ r\gamma^5\gamma^0 - \left( \omega - \frac{eV}{2} \right) \gamma^5\gamma^1 \right]$$

$$\vec{\sigma} = \frac{i}{(\mathbf{k}^2 + m^2)^{1/2}} \left[ -r\gamma^5\gamma^0 - \left( \omega - \frac{eV}{2} \right) \gamma^5\gamma^1 \right].$$

For each of the  $P$ 's, trace  $P = 2$ ,  $P^2 = P$ ; and for each of the  $\sigma$ 's, trace  $\sigma = 0$ ,  $\sigma^2 = 1$ . Therefore the operators  $P$ ,  $(1 - P)$ , and  $(1 \pm \sigma)/2$  are projection operators. A short calculation shows that each  $P$  commutes with the corresponding  $\sigma$  (i.e.,  $[\vec{P}, \vec{\sigma}] = 0$ ,  $[\vec{P}_R, \vec{\sigma}_R] = 0$ , etc.). Now define

$$\vec{\chi}_1 = \vec{n}_1 \vec{P} \left( \frac{1 + \vec{\sigma}}{2} \right) \chi_1$$

$$\vec{\chi}_2 = \vec{n}_2 \vec{P} \left( \frac{1 - \vec{\sigma}}{2} \right) \chi_2$$

$$\vec{\chi}_3 = \vec{n}_3 (1 - \vec{P}) \left( \frac{1 - \vec{\sigma}}{2} \right) \chi_3$$

$$\vec{\chi}_4 = \vec{n}_4 (1 - \vec{P}) \left( \frac{1 + \vec{\sigma}}{2} \right) \chi_4.$$

Reflected and transmitted left spinors  $\vec{\chi}_{iR}$ ,  $\vec{\chi}_{iT}$ , for  $i = 1, 4$ , and all of the right spinors  $\vec{\chi}_i$ ,  $\vec{\chi}_{iR}$ ,  $\vec{\chi}_{iT}$ , for  $i = 1, 4$ , can be defined analogously using the appropriate  $P$ 's and  $\sigma$ 's. A straightforward but tedious calculation shows that the choice

$$\vec{n}_i = \vec{n}_{iR} = \vec{n}_{iT} = \left\{ \frac{1}{2m} \left[ \frac{m}{(\mathbf{k}^2 + m^2)^{1/2}} + 1 \right] \left| \omega + \frac{eV}{2} + (\mathbf{k}^2 + m^2)^{1/2} \right| \right\}^{-1/2}$$

$$\vec{n}_i = \vec{n}_{iR} = \vec{n}_{iT} = \left\{ \frac{1}{2m} \left[ \frac{m}{(\mathbf{k}^2 + m^2)^{1/2}} + 1 \right] \left| \omega - \frac{eV}{2} + (\mathbf{k}^2 + m^2)^{1/2} \right| \right\}^{-1/2}$$

normalizes the spinors such that

$$\begin{aligned}
 i\tilde{\chi}_i^\dagger \gamma^0 \tilde{\chi}_i &= i\tilde{\chi}_{iR}^\dagger \gamma^0 \tilde{\chi}_{iR} = i\tilde{\chi}_{iT}^\dagger \gamma^0 \tilde{\chi}_{iT} = \pm \varepsilon \delta_{i,j} && \text{for } q \text{ or } r \text{ real} \\
 i\tilde{\chi}_i^\dagger \gamma^0 \tilde{\chi}_i &= i\tilde{\chi}_{iR}^\dagger \gamma^0 \tilde{\chi}_{iR} = i\tilde{\chi}_{iT}^\dagger \gamma^0 \tilde{\chi}_{iT} = \pm \varepsilon' \delta_{i,j} && \text{for } q \text{ or } r \text{ real} \\
 i\tilde{\chi}_{iR}^\dagger \gamma^0 \tilde{\chi}_i &= \pm \varepsilon' \delta_{i,j} && \text{for } r \text{ imaginary} \\
 i\tilde{\chi}_{iR}^\dagger \gamma^0 \tilde{\chi}_i &= \pm \varepsilon \delta_{i,j} && \text{for } q \text{ imaginary,}
 \end{aligned}$$

where the + sign holds for  $i = 1, 2$ , the minus sign holds for  $i = 3, 4$ , and  $\varepsilon$  and  $\varepsilon'$  are defined in Eqs. (7). It is straightforward to verify that the spinors thus defined for  $i = 1, 2$  are the spinors which appeared in the asymptotic solution Eqs. (11) of the dynamical equation. The spinors for  $i = 3, 4$  would appear in the analogous general solution of the dynamical equation under the substitution  $m \rightarrow -m$ . Note that, as might be expected,

$$\begin{aligned}
 \tilde{\chi}_{iR} &= \tilde{\chi}_{iT} \\
 \tilde{\chi}_{iR} &= \tilde{\chi}_{iT} \cdot
 \end{aligned}$$

For convenience choose a representation such that  $\gamma^1$  is real and  $\gamma^0, \gamma^2, \gamma^3$  are imaginary. In this case

$$\begin{aligned}
 P_0^* &= P_0 \\
 \sigma_0^* &= \sigma_0
 \end{aligned}$$

which implies

$$\begin{aligned}
 \chi_1^* &= \chi_2 \\
 \chi_3^* &= \chi_4
 \end{aligned}$$

and

$$\begin{aligned}
 \vec{P}^* &= \vec{P}_R \\
 \vec{\sigma}_R^* &= -\vec{\sigma}_R
 \end{aligned}$$

which implies

$$\begin{aligned}
 \tilde{\chi}_1^* &= \tilde{\chi}_{2R} \\
 \tilde{\chi}_2^* &= \tilde{\chi}_{1R} \\
 \tilde{\chi}_3^* &= \tilde{\chi}_{4R} \\
 \tilde{\chi}_4^* &= \tilde{\chi}_{3R} \cdot
 \end{aligned}$$

Similar relations hold for the right spinors.

Using the anticommutation relations on the Dirac matrices it is easy to prove the following useful identities.

Regions I, III, V

$$\begin{aligned}\vec{P}^\dagger \vec{P} &= \frac{i}{m} \gamma^0 \left( \omega + \frac{eV}{2} \right) \vec{P} \\ \vec{P}^\dagger \gamma^0 \gamma^1 \vec{P} &= -\frac{qi}{m} \gamma^0 \vec{P} \\ \vec{P}^\dagger \gamma^0 \gamma^1 \vec{P}_R &= 0.\end{aligned}$$

Region II

$$\vec{P}_T^\dagger \gamma^0 \gamma^1 \vec{P}_T = 0.$$

Region IV

$$\vec{P}_T^\dagger \gamma^0 \gamma^1 \vec{P}_T = 0$$

with analogous relations for the other energy projection operators. These relations are used extensively in the establishment of the Wronskian relations of Table II.

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