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LETTER TO THE EDITOR

High-overtone normal modes of Schwarzschild black holes

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Abstract. The frequencies of normal-mode oscillations of Schwarzschild black holes are studied using a higher-order WKB approach. The method is mathematically equivalent to locating the poles in the transmission amplitude for one-dimensional quantum mechanical tunnelling through a potential barrier. The transmission amplitude is expressed in terms of contour integrals of WKB functions around turning points of the effective barrier in the Regge–Wheeler equation. In the limit of large overtone number n, the dimensionless frequency $\sigma \equiv \omega M$ ($M =$ black-hole mass) for $l = 2$ gravitational modes is given by $\sigma = -\mathrm{i}(2n+1)/8 + \frac{1}{2}A(1+\mathrm{i})(2n+1)^{-1/2} + \ldots$, where $A \simeq 1.0$. We compare our results with numerical computations of Leaver and find striking and unexplained differences.

The normal-mode oscillations of black holes have been an area of active research for several years [1–6]. These free oscillations in the spacetime surrounding a black hole may be excited by a passing star, by the accretion of matter, or by some external perturbing field. Once the disturbance stops, the normal-mode oscillations give rise to radiation escaping to infinity, and radiation falling across the event horizon. These characteristic boundary conditions lead to a discrete spectrum of normal-mode oscillations in general.

One reason for the interest in normal modes is that they could be an important observable source of gravitational radiation [7, 8]. If gravity waves were detected from a black hole, an analysis of the spectrum in terms of normal-mode frequencies could give information about the black hole and about the physical generation of the waves. In addition, the study of normal modes is an integral part of the general question of the stability of black-hole solutions of Einstein's equations [9]. Although only the fundamental and lowest overtone modes are likely to be significantly excited in an astrophysically relevant situation [7], a knowledge of the entire spectrum may be important in understanding such questions as the stability or the thermodynamics of black holes [10].

Most of the previous work dealing with the Schwarzschild black hole, however, has dealt with only the fundamental, or low overtone modes. Chandrasekhar and Detweiler [1] solved for the frequencies by determining numerically the values which would lead to the appropriate boundary conditions. In a series of papers, Schutz, Will, Iyer and Kokkotas [3–5] used the WKB approximation to solve for the low modes. Leaver [6] used a method of solving a continued fraction eigenvalue equation to yield the frequencies for mode numbers up to $n \simeq 60$. The method outlined in this letter shows how to use the WKB approximation for modes of any order; in the limit of large n, it yields an asymptotic, analytic formula for the normal-mode frequencies.

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The standard approach to black-hole perturbation theory begins by assuming a Schwarzschild background metric, then, for the case of gravitational modes, adding a small perturbation to the metric. For modes corresponding to oscillations of test electromagnetic or scalar fields, the equations of the field are solved on the fixed Schwarzschild background. Except for the scalar field case, these field equations lead to a system of coupled differential equations. By combining these equations appropriately and separating variables, a 'master' radial equation can be derived in terms of Ψ, a function of the metric perturbations or the external fields. This is called the Regge–Wheeler equation,

$$\frac{d^2\Psi}{dr_*^2}+\left[\omega^2-\left(1-\frac{2M}{r}\right)\left(\frac{\lambda}{r^2}+\frac{2\beta M}{r^3}\right)\right]\Psi=0 \tag{1}$$

where r is the standard radial coordinate, r_* is the Regge–Wheeler 'tortoise coordinate', M is the black-hole mass, ω is the frequency, $\lambda=l(l+1)$ where l is the angular harmonic index and $\beta=-3$, 0, 1, and distinguishes the various types of external perturbations, gravitational, electromagnetic or scalar fields, respectively.

Choosing $\beta=-3$ for gravitational perturbations, $\lambda=6$ $(l=2)$, and scaling the distances r and r_* by M, with $\sigma\equiv M\omega$, we put the radial equation into the form

$$d^2\Psi/dr_*^2+Q(r)\Psi=0 \tag{2}$$

where

$$Q(r)=\sigma^2-6/r^2+18/r^3-12/r^4. \tag{3}$$

This equation for Ψ has the same form as Schrödinger's equation with $E=\sigma^2$ and $V_{\text{eff}}=6/r^2-18/r^3+12/r^4$.

The normal-mode boundary conditions which we have stated for Ψ, namely radiation escaping to $r_*=\infty$ $(r=\infty)$ and $r_*=-\infty$ $(r=2)$, and no incoming waves, are equivalent to the conditions for quantum-mechanical scattering resonances of potential barrier tunnelling. These correspond to the energies where the transmission amplitude goes to infinity.

The WKB approximation yields the asymptotic form for the tunnelling transmission amplitude [11]

$$T=(1+e^{2S})^{-1} \tag{4a}$$

$$S=\sum_{n=0}^{\infty}\oint S'_{2n}\,dr_* \tag{4b}$$

where S'_{2n} denote the even-order WKB expansion functions, given by

$$S'_0=(-Q)^{1/2} \tag{5a}$$

$$S'_2=-\tfrac{1}{8}Q''/(-Q)^{3/2}-\tfrac{5}{32}Q'^2/(-Q)^{5/2} \tag{5b}$$

and so on, and where the contours for the integrals surround the classical turning points of the effective potential V_{eff}, which are zeros of Q. For detailed discussion of the WKB approximation, see for example [12]; for an application to sub-barrier tunnelling, see [13].

The resonance condition, $T\to\infty$, then becomes an eigenvalue condition on S,

$$S=i\pi(n+\tfrac{1}{2}). \tag{6}$$

Since S is a function of σ, this yields a discrete set of normal-mode frequencies. However, for a general value of σ^2, there are four roots of Q since it is a quartic polynomial in r^{-1} (except for the case $\beta = 0$). To determine which two are the turning points to be enclosed, we first choose a value for σ^2 near the peak of V_{eff}, which gives two physical turning points r_3 and r_4, and two unphysical ones r_1 and r_2, elsewhere in the complex plane (figure 1). Such a value of σ approximates a fundamental or low-lying normal mode [3]. By continuously varying σ to arbitrary values, we follow the movement of the roots, and keep track of the turning points to be enclosed.

Numerical work by Leaver [6] has shown that the frequencies of the high-n modes have a large negative imaginary component, and a relatively small real component. Choosing a representative form for σ of $\sigma = 0.3 - ib$ and varying b from 0 to ∞, we find that the roots move in a manner shown schematically in figure 1. The branch cuts arise because of the fractional powers of Q in the integrands S'_{2n}. The WKB approach can be implemented by integrating the terms in (4b) on any contour enclosing r_3 and r_4 and the branch cut shown in figure 2, and excluding the pole at $r = 2$ (which arises from converting the integral from $\mathrm{d}r_*$ to $(1-2/r)^{-1}\,\mathrm{d}r$). The contour C_0 shown in figure 2 would be suitable, for example.

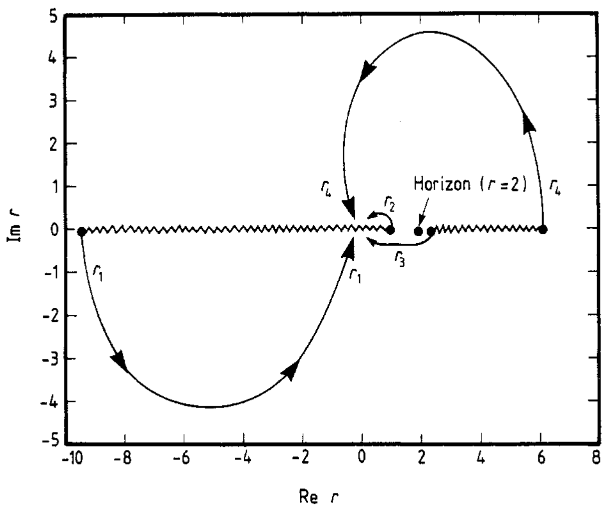

Figure 1. Migration of roots of $Q(r)$ in the complex r plane as σ varies. For small $\sigma \simeq 0.3$, roots lie on the real axis, with branch cuts joining r_1 to r_2 and r_3 to r_4. The horizon is at $r = 2$. As σ varies with a growing negative imaginary part, the roots migrate schematically as shown, heading toward $r = 0$ as $|\sigma| \to \infty$.

We have used the WKB approximation in two distinct ways to calculate high overtone frequencies. The first method seeks an analytic approximation for large n. As the magnitude of σ increases, the roots can be solved as an expansion in terms of $\sigma^{-1/2}$. To leading order, solving $Q = 0$ yields

$$r_i \simeq (12/\sigma^2)^{1/4} \tag{7}$$

so the roots approach the corners of a square as $|\sigma| \to \infty$. In the limit of large $|\sigma|$, the contour C_0 may be distorted as shown in figure 2. Thus we only need to evaluate the integrals around the contours C_1, C_2 and C_3.

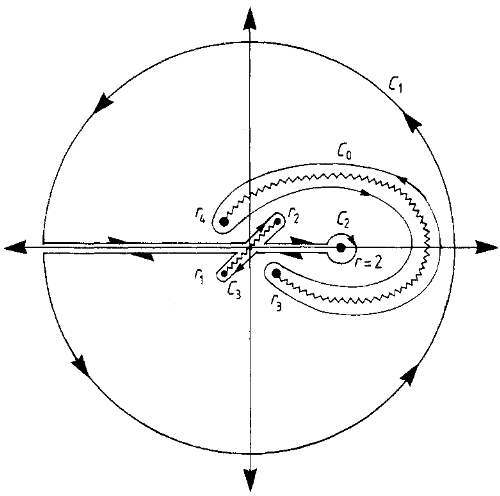

Figure 2. Contour C_0 enclosing roots r_3 and r_4. For analytic approximation for large n, contour is deformed to contours C_1, C_2 and C_3.

Consider the contours C_1 and C_2. The integral of S_0' around each yields exactly $-2\pi\sigma$, while the integrals of all the other S_{2n}' yield zero. Thus we have

$$S = -4\pi\sigma + \sum_{n=0}^{\infty} \oint_{C_3} S_{2n}'(1-2/r)^{-1}\,\mathrm{d}r. \tag{8}$$

To calculate the integrals around C_3, we factor Q in terms of the four roots,

$$Q = \sigma^2 r^{-4}(r-r_1)(r-r_2)(r-r_3)(r-r_4). \tag{9}$$

By introducing an expansion parameter η, such that $r_1 \to \eta r_1$ and $r_2 \to \eta r_2$ and expanding the integrand in a Taylor series in η about $\eta = 0$, we effectively move the roots to be encircled by C_3, r_1 and r_2, to the origin, allowing the contour integrals to be evaluated simply using a Laurent series that is valid in an annulus around the origin that excludes r_3 and r_4. Then by setting $\eta = 1$, we have an analytic series solution for the integrals in terms of the roots r_1, r_2, r_3 and r_4. We then substitute expansions for the roots in powers of $\sigma^{-1/2}$ (cf (7) for the leading term) into the series solution for S. Apart from the initial term (8), this yields S in terms of odd powers of $\sigma^{-1/2}$ only. To achieve some measure of accuracy, we expanded the integrands of S_0', S_2' and S_4' to order η^{25}, and the roots r_i to order σ^{-2} (we used the algebraic program MACSYMA). The integrals of S_2' and S_4' lead to divergent series, but by using Padé approximants we obtain reasonably convergent results. The final result is then

$$S = -4\pi\sigma + \pi(A\sigma^{-1/2} + B\sigma^{-3/2} + \ldots) \tag{10}$$

where, for $\beta = -3$, $\lambda = 6$, we find $A \simeq 1.004$, $B \simeq -0.035$, both figures good to ± 0.03 (using the optimal asymptotic approximation).

Applying the eigenvalue condition (6) allows us to invert the expansion to find σ in terms of n,

$$\sigma = -\tfrac{1}{8}\mathrm{i}(2n+1) + \tfrac{1}{2}A(1+\mathrm{i})(2n+1)^{-1/2} + 4B(\mathrm{i}-1)(2n+1)^{-3/2}$$
$$+ 2A^2(2n+1)^{-2} + \ldots. \tag{11}$$

In the second method, we use only the first-order WKB approximation, but evaluate the appropriate contour integrals numerically. We choose a trial frequency, determine the two roots to be enclosed, select an appropriate polygonal contour, and evaluate $\oint S_0' \, \mathrm{d}r_*$. The resulting complex numbers are plotted as a function of trial frequencies; those frequencies for which the real part of the contour integral vanishes and the imaginary part equals $\pi(n+\frac{1}{2})$ for some n corresponds to normal modes.

Method I is of higher WKB order, and therefore intrinsically more accurate, but the analytic approximations involved in arriving at (10) rely on the roots being close to the origin in figure 2, which is a better approximation as n increases. In method II, the evaluation of the WKB integral is just as accurate for small n as for large, but it is only first-order in WKB. We would therefore expect that method I would be better for large n, but method II may provide more accuracy for intermediate values of n. For small $n = 0, 1, 2$, the higher-order WKB method of Iyer [4] is expected to provide the best accuracy. In table 1, we give selected normal-mode frequencies evaluated using the WKB approximation: modes 0–2 are from Iyer [4], modes between 3 and 20 are from method II, and the remaining modes are from method I. For the lowest three modes, method II agrees with Iyer within 3–4%, while Iyer agreed with Leaver [6] for these modes within fractions of a percent [4]. For the large-n modes, methods I and II agree within 1% at $n = 10$, improving to 0.04% at $n = 60$, and the differences can be shown to be due to the higher-order WKB terms included in method I.

Table 1. WKB normal modes for gravitational perturbations ($l = 2$).

Mode number	Re(σ)	Im(σ)	Method
0	0.3732	−0.0892	Iyer
1	0.3460	−0.2749	
2	0.3029	−0.4711	
3	0.2676	−0.7188	Method II
4	0.2274	−0.9671	
6	0.1770	−1.4809	
8	0.1490	−1.9952	
10	0.1312	−2.5067	
15	0.1050	−3.7763	
20	0.0802	−5.0471	Method I
25	0.0715	−6.3051	
30	0.0651	−7.5610	
35	0.0602	−8.8156	
40	0.0563	−10.0694	
45	0.0530	−11.3225	
50	0.0503	−12.5752	
55	0.0479	−13.8275	
60	0.0459	−15.0795	

Figure 3 compares the frequencies obtained by these two methods with the values found by Leaver [6]. There is excellent agreement throughout on the imaginary part of the frequency, from 2% at $n = 10$ to 0.1% at $n = 58$, but there are systematic disagreements between Leaver and the WKB results on the real part. It is not clear what the origin of the discrepancy is. The most striking difference is the absence of Leaver's pure imaginary frequency mode ($n = 8$), which Chandrasekhar [14] has denoted an 'algebraically special' mode. If Leaver's mode is correct, then the WKB

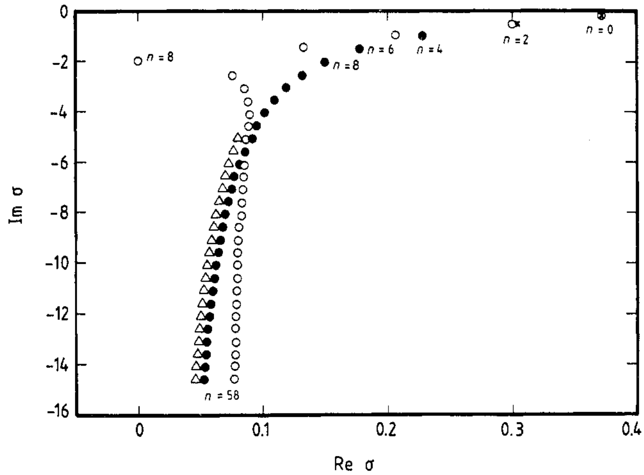

Figure 3. WKB plotted against Leaver normal modes. Only even-n modes are shown, with fundamental mode starting at $n=0$. WKB modes are denoted by: △, method I, $20 \leqslant n \leqslant 58$; ●, method II, $4 \leqslant n \leqslant 58$; ×, Iyer, $n=0, 2$. Modes found by Leaver are denoted by open circles. Leaver's mode at $\sigma \simeq -2\mathrm{i}(n=8)$ is 'algebraically special' value of Chandrasekhar [14].

method must be in error by 7% at this point; however, the difference between method I and method II is only 1%. In fact, numerical evaluation within method II of the second-order WKB terms tends to move the eigenfrequencies so obtained in the region around $n=8$ to within 0.03% of those of method I.

The other significant disagreement is the deviation of the real parts in the limit of large n. According to the WKB method, the real part of the frequency tends to zero as $(n+\frac{1}{2})^{-1/2}$, while Leaver's frequencies tend to a constant non-zero real part. At $n=58$, the difference is over five times the difference between the two WKB methods (0.04%). Again, inclusion of second-order WKB terms in method II brings the two WKB results within 0.007% of each other.

The possibility that Leaver's calculation gives wrong values for the real parts of the frequencies must be considered, especially in view of the considerable delicacy of the numerical techniques he used to evaluate his continued fractions; on the other hand, we have looked at this and have not found any obvious flaws. These questions urgently need to be clarified, because of the potentially wide applicability of the WKB method to other problems, such as higher-l modes of Schwarzschild, modes of other wave fields, modes of the Kerr metric, and applications in nuclear physics [13].

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