

Activity 16: Quadratic Equations

1. The quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, gives all of the roots (i.e. solutions) of a general quadratic equation $ax^2 + bx + c = 0$. Quadratic equations arise in many different situations in math and the sciences, but here we'll look at some general properties of the quadratic formula, without a specific application in mind.
 - a) The quadratic formula can be written as $x = v \pm \delta$, where $v = \frac{-b}{2a}$ is called the *vertex* and $\delta = \frac{\sqrt{\Delta}}{2a}$ is related to a quantity Δ - called the *discriminant* - which is given by $\Delta = b^2 - 4ac$. The interpretation of v is the x coordinate of the highest/lowest point of the parabola, and the interpretation of δ is the distance between the vertex and each of the solutions. For each of the following equations, use the quadratic formula to find the equation's roots (if any) and associated values of v and Δ .
 - i. $x^2 - 2x - 3 = 0$
 - ii. $-x^2 + 2x - 2 = 0$
 - iii. $\frac{1}{2}x^2 = 0$

b) On graph paper, make a separate graph of each of the above equations from $x = -2$ to $x = 3$. You don't have to be incredibly precise with your graphs, but you should make sure to label the roots (where the function crosses the x axis) and the vertex of each parabola. Feel free to use a graphing calculator if that is helpful. Based on your graphs, and the value of Δ associated to each one, write down a guess below about the connection between the sign of Δ (whether it is positive, negative, or zero) and the number of solutions to the associated quadratic equation.

2. A ball that is thrown up in the air at time $t = 0$ has a height at different times described by $y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$, where v_0 is the initial speed of the ball, y_0 is its initial height, and $g = 9.8\text{m/s}^2$ describes the acceleration due to the gravity of the Earth.

a) Find the time at which the ball reaches its maximum height, as well as what that height is. Explain your method. (Some of the concepts from question 1 might help here)

b) Suppose we are standing at ground level ($y = 0$) and throw the ball straight up with a speed of 40 miles per hour. First picture this situation in your head. Roughly how long do you guess it would take for the ball to hit the ground? 1 second? 5 seconds? A minute? After having made your guess, use the above formula for $y(t)$ to solve for this time. The answer should roughly agree with your intuition.

c) We now change the above situation so that we are standing on a 20 ft high platform when we throw the ball. If we throw the ball at $t = 0$ with the same speed as before, solve for the time at which the ball hits the ground. You will find that the quadratic formula, as in many situations, gives us two solutions. Explain why you chose the solution you did. Even though you rejected the other solution, can you think of a reasonable interpretation for it? (Hint: Imagine that someone else saw the ball immediately after it had been pitched, but didn't realize that you had just thrown it)

3. The solutions to the equation $(y - 1)^2 + x^2 = 2^2 = 4$ describe a circle with its center at $(x = 0, y = 1)$, and with a radius of 2. This equation has *two* independent variables, whereas the quadratic equations we normally work with only have one. We can still use the quadratic equation, though, by treating y as a variable and x as a “constant”. This process will be explained below.

a) First, use algebra to rewrite the above equation into the form $ay^2 + by + c = 0$, where a , b , and c are allowed to be functions of x . Show your work and your result below. Although x and y are both varying over the circle, our rewriting of the equation into the above form is what allows us to act like the equation only has one variable, and thereby use the quadratic formula.

- b) Having your equation in the above form, use the quadratic equation to solve for y in terms of a , b , and c . Since some of these “constants” actually depend on x , this gives us y as a function of x . On graph paper, sketch your two solutions *on the same graph*. Use separate colors to distinguish the two solutions. What does the combination of your two graphs give you? (Feel free to use a graphing calculator for this)
- c) (Bonus) The equation we started with, $(y - 1)^2 + x^2 = 4$, has two variables, and so in part a we could have chosen to rewrite it in terms of x , i.e. $ax^2 + bx + c = 0$, where a , b , and c can now be functions of y . Do this, and then, as above, use the quadratic equation to get two solutions that give you x as a function of y . On graph paper, draw these solutions on the same graph, using different colors for each solution. Does this give you the same shape as in part b? What’s the difference between the two solutions you obtained here and the two you obtained there?

(Note that, even though x is the dependent variable and y is the independent variable here, you should still draw your graphs with the same orientation of the x and y axes as you normally use (x -axis horizontal, y -axis vertical). This is not the normal convention for drawing graphs, but here it allows you to more easily compare your solutions to what you got in part b.)