

Vectors

[Philosophy: do a lot of work before even introducing components. Have the students work out for themselves, using their trig, how to find components and angles.]

A vector is an abstract mathematical entity that possesses a direction and a magnitude. The simplest physical example of a vector quantity is *displacement* - motion from one place to another. For example, if I drive 10 miles west, my *displacement* from my starting position is just that: “10 miles” (the magnitude) “west” (the direction.)

Importantly, but perhaps surprisingly, a vector **doesn't have** a position! We often draw arrows to represent vectors, but keep in mind that a vector is not an arrow, since an arrow actually has a position. Therefore, you are free to draw an arrow representing a vector **anywhere** you like. You should draw the arrow so that it makes your calculations easier, of course. And if you first put it in a place you don't like, you can cross it out and draw it elsewhere.

(You may be tempted to think that displacement vectors should always be “positioned” where the displacement starts, but in some calculations you may want to position them elsewhere, and you are free to do so.)

We indicate that a mathematical quantity is a vector by putting an arrow over it: \vec{A}

Practice Drawing Vectors

1. Draw a vector that represents a displacement 10 m NE. Position the tail of the arrow at the origin. Then draw a vector that represents a displacement 7 m N. Position the tail of this arrow at the tip of the previous arrow.

With the displacement vectors positioned this way, your drawing could represent a displacement of 10 m NE followed by a further displacement 7 m N. Do you see why?

If you make these successive displacements, your total displacement is from the origin to the tip of the second arrow. We say that the sum of the two displacements is from the tail of the first, to the tip of the second, when the two displacements are drawn “tip to tail”.

Using your drawing and the Pythagorean theorem, calculate the total displacement. You need to give both the magnitude, and the direction. The direction could be given as an angle with respect to North (for example, 25° East of North).

Now draw a displacement 7 m N, positioning the tail at the origin. This displacement is followed by a displacement 10 m NE – draw this displacement “tip to tail”. Where do you end up? You have just graphically demonstrated that

$\vec{A} + \vec{B} = \vec{B} + \vec{A}$, in other words, vector addition is commutative.

Put \vec{A} s and \vec{B} s on your diagram to label the vectors. Note that there are two arrows both representing the same vector \vec{A} and two other arrows both representing \vec{B} .

2. Two people begin walking from the origin. Each holds an end of a rope that is 141 m long. They walk at equal speeds. One walks E, the other N; after they each walk 100 m, the rope is taut and they stop. Draw arrows representing their displacements from the origin. You may want to put the tails at the origin. That's fine *for now*.

When they stop, what is the displacement of the center of the rope? The center is halfway between them – in other words, its displacement is the average of their displacements.

$$\vec{R} = \frac{\vec{A} + \vec{B}}{2}$$

This is just the usual formula for an average, but with our new vector symbols. What does it mean? To multiply a vector by a number (here we are multiplying $\vec{A} + \vec{B}$ by $\frac{1}{2}$), multiply the magnitude by the number and keep the direction the same.

On your drawing, indicate the vector sum $\vec{A} + \vec{B}$ (by redrawing one of the vectors so they are tip to tail), and the vector \vec{R} representing the displacement of the center of the rope. Check that \vec{R} really is the average of the two displacements.

3. Here's a more difficult rope question. Suppose each walker holds the end of a rope that is 300 m long. They walk at equal speeds. One walks E, the other NW. When the rope gets taut, where is the center? You may have to use some trigonometry on this one!

4. Ann walks her dog 800 m East from the origin. Bob walks 400 m Southwest. The dog then breaks free and runs straight to Bob. What distance and direction does the dog run? First sketch the vectors. Then, note that the displacement of the dog, while leashed (which is the same as Ann's displacement), plus the displacement of the run, is equal to Bob's displacement.

$$\vec{A} + \vec{D} = \vec{B}$$

We subtract \vec{A} from both sides to find \vec{D} :

$$\vec{D} = \vec{B} - \vec{A} = \vec{B} + (-1)\vec{A}$$

Sketch $(-1)\vec{A}$. Given you are planning to add it to B, where should you position it? Use trigonometry to find the distance and direction of the dog's run.

Specifying Directions & Coordinate Systems

The fact that vectors have a direction means we need a way to specify it. In two dimensions (like on a plane), we really only need a directional axis. For example, we could agree to measure all angles with respect to due North, with positive angles representing a counter clockwise rotation. For example, North-by-Northwest would be $+22.5^\circ$ (and a very fine movie, to boot!)

5. Using this convention, sketch a displacement of 200 m at 60° , followed by a displacement of 400 m at 200° . What is the total displacement? (You may want to review the laws of sines and cosines for this!)

Coordinate Systems – An easier way!

You will have noticed that the problems above are “not trivial”! They take considerable thought, sketching, and a hefty application of your trigonometric tools like the laws of sines and cosines. There is an easier way to do vector computations: use a coordinate system and two axes (in a plane; three are needed in 3D.)

For now, let's take our $+x$ -axis to East and our $+y$ -axis to the North. We can get anywhere in the plane by making a displacement East (possibly a negative East, which is West), followed by (in other words, plus) a displacement North (possibly a negative North, which is South). Thus, any displacement in the plane can be written as the sum of an Eastward (x) displacement plus a Northward (y) displacement.

6a. What Eastward and Northward displacements sum to give a displacement of 200 m Southeast?

6b. What Eastward and Northward displacements sum to give a displacement 200 m 60° East of North?

6c. What eastward and northward displacements sum to give a displacement 200 m 20° West of North?

The magnitudes of the x and y displacements are called “components” of a vector. (Sometimes physicists get sloppy and call the x and y displacements components... it really doesn't matter much what you call things.) The easiest way to add two or more vectors is to add their x components together, and add their y -components together, to get the x and y components of the sum. You can see why this works: vector addition is commutative and associative, so you can add the vector components in any order you like. Adding all the x components together is just easiest, because they add as scalars (simple numbers.)

The first step to solving any vector problem using this approach is to find the components of every vector in the problem.

7a. What are the x and y components of Ann's and Bob's displacements?

7b. What are the x and y components of a displacement 200 m that makes an angle of 60° left of the y -axis?

To find a vector sum, add the components. To subtract vectors, subtract components.

7c. What are the x and y components of the dog's run?

The use of components is so pervasive that we often simply give a vector in terms of its components... in other words, the direction of the vector is not explicitly given. When we give a vector in terms of its components, we need a formal way to show that is what we are doing. To accomplish this, we use “unit vectors,” which are vectors 1 “unit” in magnitude in the x and y directions. We often use a “hat” over an i for the x-direction, over a j for the y direction, and over a k for the z direction.

By the way, “unit vectors” have no units. So if you want to write a displacement of 200 meters in the +x direction, you write

$$200\hat{i} \text{ meters}$$

The unit vector is only there to show the direction, not the units or magnitude.

Often, though, you’ll want to know the direction of a vector, not just its components. You can always figure out the angle a vector makes with the x or y axis by using your trigonometry.

8. What angle do each of the following vectors make with the x-axis?

$$5\hat{i} + 8\hat{j}$$

$$12\hat{i} - 20\hat{j}$$

$$-4\hat{i} - 6\hat{j}$$

What quadrant is each in? Sketch each quickly.

What angle do each of the following make with the y-axis?

$$6\hat{i} + 3\hat{j}$$

$$10\hat{i} - 2\hat{j}$$

$$-\hat{i} - 6\hat{j}$$

What quadrant is each in? Sketch each quickly.

There’s nothing special about the axis directions. You can choose any direction you like to be your x-axis. Of course, once you’ve chosen your x-axis, your y-axis must be perpendicular to it. (It’s usual to go counterclockwise to get from +x to +y, but you don’t have to follow this convention if you don’t want to. It just gets mighty confusing if you make bigger numbers on the y axis as you down, instead of up.)

9. Consider a block sitting on a slide. The slide is 30° from horizontal. The weight of the block is the force of the Earth pulling it straight down. Forces are vectors, as they have a magnitude and a direction. Draw a sketch and then draw an x-axis that is parallel to the slide (not horizontal.) If the magnitude of the weight is W, what is the (vector) weight in your xy coordinate system?

10. For an object to remain stationary, it is necessary (but not sufficient!) that all the forces on the object add (vectorially) to zero. This means that all the x-components must sum to zero and all the y-components must sum to zero. If the block in the previous problem is held on the slide by a rope that is parallel to the slide, what is the tension (force) exerted by the rope on the block? Add a rope to your sketch. (Assume the slide is very slippery, so that no friction helps the block stay put.)

11. A pendulum is a weight, like a lead bob, hanging on a rope that can swing back and forth. Although the bob may be moving, it only moves along part of a circle; the circle has its center where the rope hangs from. Do you see why?

Consider an instant in time when the bob reaches its highest point in its swing, so that the rope makes an angle of θ with respect to the vertical. Make a sketch. On your sketch, draw vectors that represent the weight of the bob and the tension force that the rope may exert on the bob.

Remember that the bob has to move along a circular path. Draw a coordinate system on the bob in your sketch. Choose your coordinate system so that the bob can only move along either the x-axis or the y-axis. (Of course, at the instant we are concerned with, the bob is not moving. But it will start swinging downward in an instant!)

Although the bob begins to swing downward, it remains stationary along one of your axes. (This is a difficult concept: it “remains stationary” along this axis only for an instant! – possibly we should use this example after calculus!) Thus, the sum of the components of all the forces along that axis must be zero. Use this fact to find the tension force that the rope exerts on the bob as a function of θ . What happens when $\theta=90^\circ$?

12. Consider a block with weight W , held up by two ropes. The ropes make an angle of θ with respect to the vertical. Sketch the situation. How must the ropes be oriented? What is the tension force that each rope exerts on the block? What is the tension force in the limiting cases when $\theta=0^\circ$ and $\theta=90^\circ$?

Practice Problems

1. Jim’s dog Sparky runs 50 m northeast to a tree, then 70 m west, and finally 20 south. Draw a picture and choose a coordinate system. Find Sparky’s net displacement in components, and as a magnitude and angle.

2. In physics, “velocity”, like displacement, is a vector. (It’s actually displacement divided by the time interval; since time is a scalar (a simple number), you know that velocity is a vector.)

A cannon launches cannonballs with a speed of 200 m/s. (“Speed” is the magnitude of velocity.) What angle upward should the cannon be pointed if you want the horizontal component of the velocity to be 140 m/s?

3. A battery pack in the space station is held by three ropes. (In the orbiting space station, we can essentially ignore the force of gravity acting on the battery pack.) One rope pulls outward (away from Earth) with 10 units of force. A second rope pulls at an angle of 45° with respect to downward (downward = toward the Earth) with 12 units of force. Sketch the situation. If the battery pack is to remain stationary (in the space station), what is the orientation of and tension in the 3rd rope?