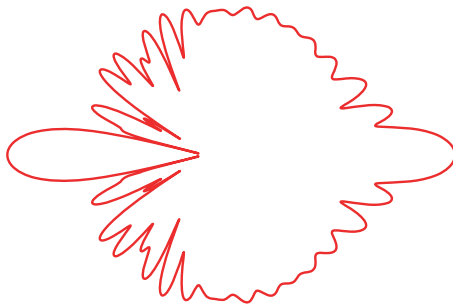
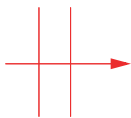


GERHARD KRISTENSSON

SPHERICAL VECTOR WAVES

January 17, 2014



Vector operations with ∇

- (1) $\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi$
- (2) $\nabla(\varphi\psi) = \psi\nabla\varphi + \varphi\nabla\psi$
- (3) $\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$
- (4) $\nabla(\mathbf{a} \cdot \mathbf{b}) = -\nabla \times (\mathbf{a} \times \mathbf{b}) + 2(\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a})$

- (5) $\nabla \cdot (\mathbf{a} + \mathbf{b}) = \nabla \cdot \mathbf{a} + \nabla \cdot \mathbf{b}$
- (6) $\nabla \cdot (\varphi\mathbf{a}) = \varphi(\nabla \cdot \mathbf{a}) + (\nabla\varphi) \cdot \mathbf{a}$
- (7) $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$

- (8) $\nabla \times (\mathbf{a} + \mathbf{b}) = \nabla \times \mathbf{a} + \nabla \times \mathbf{b}$
- (9) $\nabla \times (\varphi\mathbf{a}) = \varphi(\nabla \times \mathbf{a}) + (\nabla\varphi) \times \mathbf{a}$
- (10) $\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$
- (11) $\nabla \times (\mathbf{a} \times \mathbf{b}) = -\nabla(\mathbf{a} \cdot \mathbf{b}) + 2(\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a})$

- (12) $\nabla \cdot \nabla\varphi = \nabla^2\varphi = \Delta\varphi$
- (13) $\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}$
- (14) $\nabla \times (\nabla\varphi) = \mathbf{0}$
- (15) $\nabla \cdot (\nabla \times \mathbf{a}) = 0$
- (16) $\nabla^2(\varphi\psi) = \varphi\nabla^2\psi + \psi\nabla^2\varphi + 2\nabla\varphi \cdot \nabla\psi$

- (17) $\nabla r = \hat{\mathbf{r}}$
- (18) $\nabla \times \mathbf{r} = \mathbf{0}$
- (19) $\nabla \times \hat{\mathbf{r}} = \mathbf{0}$
- (20) $\nabla \cdot \mathbf{r} = 3$
- (21) $\nabla \cdot \hat{\mathbf{r}} = \frac{2}{r}$
- (22) $\nabla(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a}$, \mathbf{a} constant vector
- (23) $(\mathbf{a} \cdot \nabla)\mathbf{r} = \mathbf{a}$
- (24) $(\mathbf{a} \cdot \nabla)\hat{\mathbf{r}} = \frac{1}{r}(\mathbf{a} - \hat{\mathbf{r}}(\mathbf{a} \cdot \hat{\mathbf{r}})) = \frac{\mathbf{a}_\perp}{r}$
- (25) $\nabla^2(\mathbf{r} \cdot \mathbf{a}) = 2\nabla \cdot \mathbf{a} + \mathbf{r} \cdot (\nabla^2\mathbf{a})$

- (26) $\nabla u(f) = (\nabla f) \frac{du}{df}$
- (27) $\nabla \cdot \mathbf{F}(f) = (\nabla f) \cdot \frac{d\mathbf{F}}{df}$
- (28) $\nabla \times \mathbf{F}(f) = (\nabla f) \times \frac{d\mathbf{F}}{df}$
- (29) $\nabla = \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \nabla) - \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \nabla)$

Spherical Vector Waves

by

Gerhard Kristensson

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Lund, January 17, 2014

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Preface

This textbook treats some of the necessary prerequisites for the analysis of spherical vector wave solutions to the Maxwell equations. It is an excerpt of the much more detailed textbook by the author, *viz. Scattering of Electromagnetic Waves*. For convenience, the definitions of special functions used in this text and their elementary properties are collected in a series of appendices.

The course requires a certain knowledge of basic electromagnetic field theory, for instance the basic course in electromagnetic field theory at an undergraduate level. We expect the Maxwell field equations to be known, as well as basic vector analysis, and calculations with the nabla operator ∇ .

Exercises or problem are gathered at the end of each chapter. Advanced exercises are marked with a star (*). Answers to the exercises are found at the end of the book.

Prerequisites

The foundation of the electromagnetics stands on the shoulders of the scientific giants of the 19th century. Stars like André Marie Ampère¹, Michael Faraday², and James Clerk Maxwell³ shine brightly, see Figure 1.1. Many other scientist have contributed to the theory. A few of these giants are shown in Figure 1.2.

The physics of electromagnetic phenomena takes place in space and time. Therefore, a time dependent description is a natural starting point of modeling the electromagnetic interaction with matter. In fact, this approach is the guiding principle throughout the first part of this chapter, which is devoted to modeling of electromagnetic interaction with matter. By taking this viewpoint, we avoid some of the pitfalls that might occur if you start with a frequency domain formulation. In particular, causality is naturally included in the modeling.

In this chapter we review the basic equations that model electromagnetic wave propagation — the Maxwell equations — and we set the notation used in this book. This review and the most important consequences of the Maxwell equations, *e.g.*, the boundary conditions between two materials and conservation of power, are presented in Section 1.1.

1.1 The Maxwell equations

The Maxwell equations are the fundamental mathematical model for all theoretical analysis of macroscopic electromagnetic phenomena. James Clerk Maxwell realized that light is an electromagnetic disturbance, and he published this result in 1864 in a paper entitled: *A dynamical theory of the electromagnetic field* [9]. His famous equations were published almost a decade later in 1873 in his textbook: *A Treatise on Electricity and Magnetism* [10, 11].

All experimental tests performed since then have confirmed this model, and, through the years, an impressive amount of evidences for the validity of these equations have been gathered in different fields of applications. However, microscopic

¹André Marie Ampère (1775–1836), French physicist.

²Michael Faraday (1791–1867), English chemist and physicist.

³James Clerk Maxwell (1831–1879), Scottish physicist and mathematician.



Figure 1.1: The pioneers of electromagnetic theory. From left to right: André Marie Ampère (1775–1836), French physicist. Michael Faraday (1791–1867), English chemist and physicist. James Clerk Maxwell (1831–1879), Scottish physicist and mathematician.

phenomena require a more refined model including also quantum effects, but these effects are out of the scope of this treatment.

The Maxwell equations are the cornerstone in the analysis of macroscopic electromagnetic wave propagation phenomena.⁴ The Maxwell equations in SI-units (MKSA) are:

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \quad (1.1)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, t) + \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} \quad (1.2)$$

The equation (1.1) (or the corresponding integral formulation) is used to called Faraday’s law of induction, and the equation (1.2) is often called Ampère-Maxwell law. The different vector fields in the Maxwell equations are⁵:

$\mathbf{E}(\mathbf{r}, t)$	Electric field [V/m]
$\mathbf{H}(\mathbf{r}, t)$	Magnetic field [A/m]
$\mathbf{D}(\mathbf{r}, t)$	Electric flux density [As/m ²]
$\mathbf{B}(\mathbf{r}, t)$	Magnetic flux density or magnetic induction [Vs/m ²]
$\mathbf{J}(\mathbf{r}, t)$	Current density [A/m ²]

All these fields are functions of space and time, *i.e.*, space coordinates \mathbf{r} and time t . Often these arguments are suppressed. Only when the equations and the

⁴It is out of the scope of this textbook to present a derivation of these equations. Several excellent derivations of these macroscopic equations from a microscopic formulation are found in the literature, see *e.g.*, [4, 5, 13].

⁵Sometimes we will for simplicity use the names \mathbf{E} -field, \mathbf{D} -field, \mathbf{B} -field, and \mathbf{H} -field.

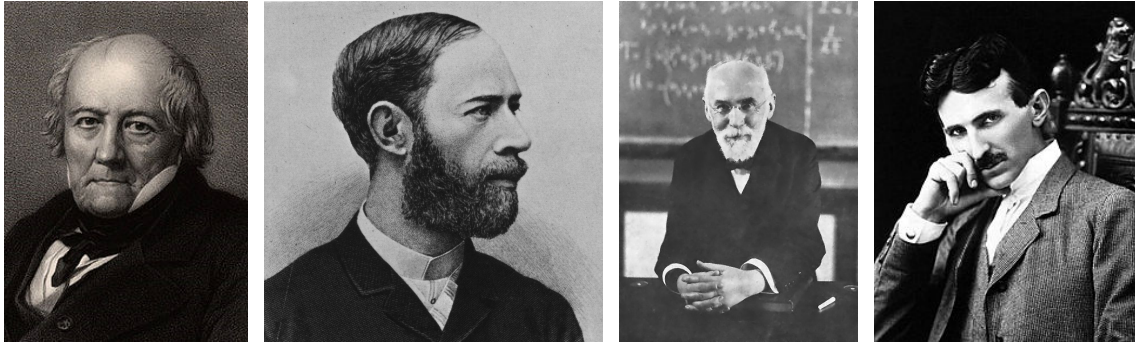


Figure 1.2: Immortal scientists of electromagnetic theory. From left to right: Jean-Baptiste Biot (1774–1862), French physicist, astronomer, and mathematician. Heinrich Rudolf Hertz (1857–1894), German physicist. Hendrik Antoon Lorentz (1853–1928), Dutch physicist. Nikola Tesla (1856–1943), Serbian inventor, mechanical engineer, and electrical engineer.

expression can be misinterpreted, we make sure the arguments are explicitly written out.

The electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic flux density $\mathbf{B}(\mathbf{r}, t)$ are defined by the force, $\mathbf{F}(t)$, on a charged particle by Lorentz' force.⁶

$$\mathbf{F}(t) = q \{ \mathbf{E}(\mathbf{r}, t) + \mathbf{v}(t) \times \mathbf{B}(\mathbf{r}, t) \} \quad (1.3)$$

where q is the electric charge of the particle located at $\mathbf{r}(t)$, and $\mathbf{v}(t)$ is its velocity.

The free charges in the material, *e.g.*, the conduction electrons, are described by the current density $\mathbf{J}(\mathbf{r}, t)$. The field contributions from bounded charges, *e.g.*, the electrons bound to the kernel of the atom, are included in the electric flux density $\mathbf{D}(\mathbf{r}, t)$.

One of the fundamental assumptions in physics is that electric charges are indestructible, *i.e.*, the sum of the charges is always constant. This invariance principle is very carefully tested. One way of expressing the conservation of charges in mathematical terms is through the continuity law of charges

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0 \quad (1.4)$$

Here $\rho(\mathbf{r}, t)$ is the charge density (charge/unit volume) that is associated with the current density $\mathbf{J}(\mathbf{r}, t)$. The charge density $\rho(\mathbf{r}, t)$ therefore models the free charges of the problem. As alluded to above, the contributions from bounded charges are included in the electric flux density $\mathbf{D}(\mathbf{r}, t)$ and the magnetic field $\mathbf{H}(\mathbf{r}, t)$.

Two additional equations are usually associated with the Maxwell equations.

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \quad (1.5)$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho(\mathbf{r}, t) \quad (1.6)$$

⁶Hendrik Antoon Lorentz (1853–1928), Dutch physicist.

Equation (1.5) tells us that no magnetic charges exist, and it implies that the magnetic flux is conserved. The equation (1.6) is usually called Gauss' law⁷. Under suitable assumptions, both these equations can be derived from the equations (1.1), (1.2) and (1.4). To see this, take the divergence of (1.1) and (1.2). This implies

$$\begin{cases} \nabla \cdot \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} = 0 \\ \nabla \cdot \mathbf{J}(\mathbf{r}, t) + \nabla \cdot \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} = 0 \end{cases}$$

since $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ for an arbitrary vector field \mathbf{A} . Interchanging the order of differentiation and using (1.4) give

$$\begin{cases} \frac{\partial(\nabla \cdot \mathbf{B}(\mathbf{r}, t))}{\partial t} = 0 \\ \frac{\partial(\nabla \cdot \mathbf{D}(\mathbf{r}, t) - \rho(\mathbf{r}, t))}{\partial t} = 0 \end{cases}$$

These equations imply

$$\begin{cases} \nabla \cdot \mathbf{B}(\mathbf{r}, t) = f_1(\mathbf{r}) \\ \nabla \cdot \mathbf{D}(\mathbf{r}, t) - \rho(\mathbf{r}, t) = f_2(\mathbf{r}) \end{cases}$$

where $f_1(\mathbf{r})$ and $f_2(\mathbf{r})$ are two functions that do not depend on time t , but can depend on the spatial coordinates \mathbf{r} . If the fields $\mathbf{B}(\mathbf{r}, t)$, $\mathbf{D}(\mathbf{r}, t)$ and $\rho(\mathbf{r}, t)$ are identically zero before a fixed time, τ , *i.e.*,

$$\begin{cases} \mathbf{B}(\mathbf{r}, t) = \mathbf{0} \\ \mathbf{D}(\mathbf{r}, t) = \mathbf{0} \\ \rho(\mathbf{r}, t) = 0 \end{cases} \quad t < \tau$$

then the equations (1.5) and (1.6) follow. Of course, static or time-harmonic fields do not satisfy this assumption, since there is no time, τ , before which all fields are zero.⁸ However, under the assumption that fields and charges do not have existed for ever, it is sufficient to use the equations (1.1), (1.2) and (1.4).

In vacuum the electric field $\mathbf{E}(\mathbf{r}, t)$ and the electric flux density $\mathbf{D}(\mathbf{r}, t)$ are parallel — the difference is in unit they are measured. The same holds for the magnetic flux density $\mathbf{B}(\mathbf{r}, t)$ and the magnetic field $\mathbf{H}(\mathbf{r}, t)$. We have

$$\begin{cases} \mathbf{D}(\mathbf{r}, t) = \epsilon_0 \mathbf{E}(\mathbf{r}, t) \\ \mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{H}(\mathbf{r}, t) \end{cases} \quad (1.7)$$

where ϵ_0 and μ_0 are the permittivity and the permeability of vacuum. Numerical values of these constants are: $\epsilon_0 \approx 8.854 \cdot 10^{-12}$ As/Vm and $\mu_0 = 4\pi \cdot 10^{-7}$ Vs/Am \approx

⁷Johann Carl Friedrich Gauss (1777–1855). German mathematician.

⁸We will return to the derivation of equations (1.5) and (1.6) for time-harmonic fields in Section 1.2 on page 10.

$1.257 \cdot 10^{-6}$ Vs/Am. The Maxwell equations in vacuum becomes

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\mu_0 \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t} \quad (1.8)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, t) + \epsilon_0 \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} \quad (1.9)$$

1.1.1 Energy conservation and Poynting's theorem

Energy conservation is shown from the Maxwell equations (1.1) and (1.2).

$$\begin{cases} \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Make a scalar multiplication of the first equation with \mathbf{H} and the second equation with \mathbf{E} and subtract. The result is

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{E} \cdot \mathbf{J} = 0$$

We rewrite this expression with the use of the differential rule of the nabla-operator $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$. We have

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{E} \cdot \mathbf{J} = 0$$

The vector product of the electric and the magnetic field plays a special role, and we introduce Poynting's vector,⁹ $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. We get Poynting's theorem.

$$\nabla \cdot \mathbf{S} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{E} \cdot \mathbf{J} = 0$$

We restrict ourselves to vacuum, $\mathbf{D} = \epsilon_0 \mathbf{E}$, and $\mathbf{B} = \mu_0 \mathbf{H}$. The Poynting theorem then reads

$$\nabla \cdot \mathbf{S} + \frac{1}{2} \frac{\partial}{\partial t} [\mu_0 |\mathbf{H}|^2 + \epsilon_0 |\mathbf{E}|^2] + \mathbf{E} \cdot \mathbf{J} = 0 \quad (1.10)$$

since $2\mathbf{E} \cdot \partial_t \mathbf{E} = \partial_t |\mathbf{E}|^2$ and $2\mathbf{H} \cdot \partial_t \mathbf{H} = \partial_t |\mathbf{H}|^2$.

Poynting's vector \mathbf{S} gives the power per unit area of the electromagnetic field or the power flow in the direction of the vector \mathbf{S} . This becomes clearer if we integrate (1.10) over a simply connected volume V , bounded by the surface S and with unit outward normal vector $\hat{\nu}$, see Figure 1.3, and use the divergence theorem. We get

$$\begin{aligned} \iint_S \mathbf{S} \cdot \hat{\nu} \, dS &= \iiint_V \nabla \cdot \mathbf{S} \, dv \\ &= -\frac{1}{2} \frac{\partial}{\partial t} \iiint_V [\mu_0 |\mathbf{H}|^2 + \epsilon_0 |\mathbf{E}|^2] \, dv - \iiint_V \mathbf{E} \cdot \mathbf{J} \, dv \end{aligned} \quad (1.11)$$

The terms are interpreted in the following way:

⁹John Henry Poynting (1852–1914), English physicist.

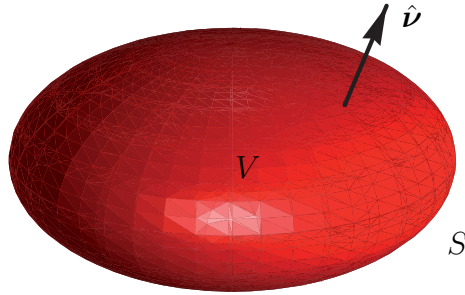


Figure 1.3: Geometry of integration.

The left-hand side:

$$\iint_S \mathbf{S} \cdot \hat{\nu} \, dS$$

This is the total power radiated out of the bounding surface S , *i.e.*, the energy per time unit, carried by the electromagnetic field.

The right-hand side: The power flow through the surface S is compensated by two different contributions. The first volume integral on the right-hand side

$$\frac{1}{2} \frac{\partial}{\partial t} \iiint_V [\mu_0 |\mathbf{H}|^2 + \epsilon_0 |\mathbf{E}|^2] \, dv$$

gives the power stored in the electromagnetic field in the volume V .

The second volume integral in (1.11)

$$\iiint_V \mathbf{E} \cdot \mathbf{J} \, dv$$

gives the work per unit time, *i.e.*, the power, that the electric field does on the charges in V .

This interpretation implies that (1.11) expresses power balance in the volume V , *i.e.*,

$$\begin{aligned} & \text{Through } S \text{ radiated power} + \text{power consumption in } V \\ & = - \text{power bounded to the electromagnetic field in } V \end{aligned}$$

1.2 Time harmonic fields and Fourier transform

Several important applications use time harmonic fields. In this section, we analyze the special simplifications time harmonic fields introduce.

We obtain the time harmonic case from the general results in the previous section by a Fourier¹⁰ transform in the time variable of all fields (dyadic-valued, vector-valued, and scalar-valued fields). We investigate the consequences time harmonic

¹⁰Jean Baptiste Joseph Fourier (1768–1830), French mathematician and physicist.

fields have on the constitutive relations and we introduce the concept of active, passive and lossless media. Moreover, the concept of reciprocity is introduced, and we investigate the polarization state of a time harmonic field, which leads to concept of the polarization ellipse.

The Fourier transform in the time variable of a vector field, *e.g.*, the electric field $\mathbf{E}(\mathbf{r}, t)$, is defined as

$$\mathbf{E}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, t) e^{i\omega t} dt$$

with its inverse transform

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t} d\omega$$

Similarly, the Fourier transform for all other time dependent fields, dyadics, and scalars are defined. To avoid heavy notation, we use the same symbol for the physical field $\mathbf{E}(\mathbf{r}, t)$, as for the Fourier transformed field $\mathbf{E}(\mathbf{r}, \omega)$ — only the argument differs. Moreover, note that the Fourier transformed field no longer has the same unit as the time domain field, *e.g.*, the physical electric field $\mathbf{E}(\mathbf{r}, t)$ has the unit V/m, but the Fourier transformed field $\mathbf{E}(\mathbf{r}, \omega)$ has the unit Vs/m.

In most cases the context suggests whether it is the physical field or the Fourier transformed field that is intended. When there is doubts which field that is intended, the time argument t or the angular frequency $\omega = 2\pi f$, where f is the frequency, is explicitly written out to distinguish the fields.

All physical quantities are real-valued, which imply constraints on the Fourier transform. The negative values of ω are related to the positive values of ω by a complex conjugate. To see this, we write down the criterion for the field \mathbf{E} to be real

$$\int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t} d\omega = \left\{ \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t} d\omega \right\}^*$$

where the star (*) denotes the complex conjugate. For real ω , we have

$$\int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t} d\omega = \int_{-\infty}^{\infty} \mathbf{E}^*(\mathbf{r}, \omega) e^{i\omega t} d\omega = \int_{-\infty}^{\infty} \mathbf{E}^*(\mathbf{r}, -\omega) e^{-i\omega t} d\omega$$

where we in the last integral has made a change of variable $\omega \rightarrow -\omega$. Therefore, for real ω we have

$$\mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}^*(\mathbf{r}, -\omega) \quad (1.12)$$

This shows that when the physical field is constructed from its Fourier transform, it suffices to integrate over the non-negative frequencies only. By a change of variable,

Time dependence $e^{-i\omega t}$ vs $e^{i\omega t}$ ($e^{j\omega t}$)

There are two sign conventions for the temporal (inverse) Fourier transform. There is the one we use in this textbook, *i.e.*, $e^{-i\omega t}$, which is used mostly by physicists. Electrical engineers often prefer the opposite sign in the exponential, *i.e.*, $e^{i\omega t}$ or $e^{j\omega t}$. The choice of sign is, of course, irrelevant in the computation of all physical quantities, but it leads to different signs in many of the complex quantities that are used in the calculations.

The choice of the electrical engineers is most appropriate when dealing with circuit applications where the dependence of the space variables is suppressed. However, using the $e^{i\omega t}$ time convention in wave propagation problems, like the scattering problems we are dealing within this textbook, leads to an extra minus sign in front of the spatial dependence, *e.g.*, an outgoing spherical wave would be e^{-ikr}/kr with this time convention.

$\omega \rightarrow -\omega$, and the use of the condition (1.12), we have

$$\begin{aligned}
 \mathbf{E}(\mathbf{r}, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t} d\omega \\
 &= \frac{1}{2\pi} \int_0^{\infty} (\mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t} + \mathbf{E}(\mathbf{r}, -\omega) e^{i\omega t}) d\omega \\
 &= \frac{1}{2\pi} \int_0^{\infty} (\mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t} + \mathbf{E}^*(\mathbf{r}, \omega) e^{i\omega t}) d\omega = \frac{1}{\pi} \operatorname{Re} \int_0^{\infty} \mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t} d\omega
 \end{aligned} \tag{1.13}$$

where $\operatorname{Re} z$ denotes the real part of the complex number z . A similar result holds for all other Fourier transformed fields that we are using. We also conclude that the real part of $\mathbf{E}(\mathbf{r}, \omega)$ is an even function of ω and the imaginary part of $\mathbf{E}(\mathbf{r}, \omega)$ is an odd function of ω .

Fields that are purely time harmonic are of special interests in many applications, see Table 1.1. If we concentrate on the time dependence, a purely time harmonic fields have time dependence of the form

$$\cos(\omega_0 t - \alpha)$$

Such fields are generated by the following Fourier transform:

$$\begin{aligned}
 \mathbf{E}(\mathbf{r}, \omega) &= \pi \left\{ \delta(\omega - \omega_0) (\hat{\mathbf{x}} E_x(\mathbf{r}) + \hat{\mathbf{y}} E_y(\mathbf{r}) + \hat{\mathbf{z}} E_z(\mathbf{r})) \right. \\
 &\quad \left. + \delta(\omega + \omega_0) (\hat{\mathbf{x}} E_x^*(\mathbf{r}) + \hat{\mathbf{y}} E_y^*(\mathbf{r}) + \hat{\mathbf{z}} E_z^*(\mathbf{r})) \right\} \\
 &= \pi \left\{ \delta(\omega - \omega_0) (\hat{\mathbf{x}} |E_x(\mathbf{r})| e^{i\alpha(\mathbf{r})} + \hat{\mathbf{y}} |E_y(\mathbf{r})| e^{i\beta(\mathbf{r})} + \hat{\mathbf{z}} |E_z(\mathbf{r})| e^{i\gamma(\mathbf{r})}) \right. \\
 &\quad \left. + \delta(\omega + \omega_0) (\hat{\mathbf{x}} |E_x(\mathbf{r})| e^{-i\alpha(\mathbf{r})} + \hat{\mathbf{y}} |E_y(\mathbf{r})| e^{-i\beta(\mathbf{r})} + \hat{\mathbf{z}} |E_z(\mathbf{r})| e^{-i\gamma(\mathbf{r})}) \right\}
 \end{aligned}$$

where $\alpha(\mathbf{r})$, $\beta(\mathbf{r})$ and $\gamma(\mathbf{r})$ are the complex phase of the components, $\omega_0 \geq 0$, and where $\delta(\omega)$ denotes the delta function. Note that this Fourier transform satisfies $\mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}^*(\mathbf{r}, -\omega)$, which is the criterion for a real-valued field. The inverse Fourier transform then gives

$$\begin{aligned}
 \mathbf{E}(\mathbf{r}, t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t} d\omega \\
 &= \{ \hat{\mathbf{x}} |E_x(\mathbf{r})| \cos(\omega_0 t - \alpha(\mathbf{r})) + \hat{\mathbf{y}} |E_y(\mathbf{r})| \cos(\omega_0 t - \beta(\mathbf{r})) \\
 &\quad + \hat{\mathbf{z}} |E_z(\mathbf{r})| \cos(\omega_0 t - \gamma(\mathbf{r})) \}
 \end{aligned}$$

A simple way of obtaining purely time harmonic waves is to employ the following expression:

$$\mathbf{E}(\mathbf{r}, t) = \operatorname{Re} \{ \mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t} \} \tag{1.14}$$

where $\mathbf{E}(\mathbf{r}, \omega)$ is a complex-valued vector. If we write $\mathbf{E}(\mathbf{r}, \omega)$ as

$$\begin{aligned}
 \mathbf{E}(\mathbf{r}, \omega) &= \hat{\mathbf{x}} E_x(\mathbf{r}, \omega) + \hat{\mathbf{y}} E_y(\mathbf{r}, \omega) + \hat{\mathbf{z}} E_z(\mathbf{r}, \omega) \\
 &= \hat{\mathbf{x}} |E_x(\mathbf{r}, \omega)| e^{i\alpha(\mathbf{r})} + \hat{\mathbf{y}} |E_y(\mathbf{r}, \omega)| e^{i\beta(\mathbf{r})} + \hat{\mathbf{z}} |E_z(\mathbf{r}, \omega)| e^{i\gamma(\mathbf{r})}
 \end{aligned}$$

Band	Frequency	Wave length	Application
ELF	< 3 KHz	> 100 km	
VLF	3–30 KHz	100–10 km	Navigation
LV	30–300 KHz	10–1 km	Navigation
MV	300–3000 KHz	1000–100 m	Radio
KV (HF)	3–30 MHz	100–10 m	Radio
VHF	30–300 MHz	10–1 m	FM, TV
UHF	300–1000 MHz	100–30 cm	Radar, TV, mobile communication
† ^a	1–30 GHz	30–1 cm	Radar, satellite communication
† ^a	30–300 GHz	10–1 mm	Radar
	4.2–7.9 · 10 ¹⁴ Hz	0.38–0.72 μm	Visible light

^aSee also Table 1.2.

Table 1.1: The spectrum of the electromagnetic waves.

Band	Frequency (GHz)
L	1–2
S	2–4
C	4–8
X	8–12
K _u	12–18
K	18–27
K _a	27–40
Millimeter band	40–300

Table 1.2: Table of radar band frequencies.

we obtain the same result as in the expression above (without the index 0 on ω). This way of constructing purely time harmonic waves are convenient and often used. Note that the field $\mathbf{E}(\mathbf{r}, \omega)$ has the same unit as the field $\mathbf{E}(\mathbf{r}, t)$. This is in contrast to the Fourier transformation of the field above, but this difference seldom causes problems.

1.2.1 The Maxwell equations

As a first step in our analysis of time harmonic fields, we Fourier transform the Maxwell equations (1.1) and (1.2) ($\frac{\partial}{\partial t} \rightarrow -i\omega$)

$$\nabla \times \mathbf{E}(\mathbf{r}, \omega) = i\omega \mathbf{B}(\mathbf{r}, \omega) \quad (1.15)$$

$$\nabla \times \mathbf{H}(\mathbf{r}, \omega) = \mathbf{J}(\mathbf{r}, \omega) - i\omega \mathbf{D}(\mathbf{r}, \omega) \quad (1.16)$$

The explicit harmonic time dependence $e^{-i\omega t}$ has been suppressed from both sides of these equations, *i.e.*, the physical fields are

$$\mathbf{E}(\mathbf{r}, t) = \text{Re} \{ \mathbf{E}(\mathbf{r}, \omega) e^{-i\omega t} \}$$

This convention is applied to all purely time harmonic fields. Note that the electromagnetic fields $\mathbf{E}(\mathbf{r}, \omega)$, $\mathbf{B}(\mathbf{r}, \omega)$, $\mathbf{D}(\mathbf{r}, \omega)$ and $\mathbf{H}(\mathbf{r}, \omega)$, and the current density $\mathbf{J}(\mathbf{r}, \omega)$ in general are complex-valued vector fields.

The continuity equation (1.4) is transformed in a similar way and we have

$$\nabla \cdot \mathbf{J}(\mathbf{r}, \omega) - i\omega\rho(\mathbf{r}, \omega) = 0 \quad (1.17)$$

The remaining two equations from Section 1.1, (1.5) and (1.6), are transformed into

$$\nabla \cdot \mathbf{B}(\mathbf{r}, \omega) = 0 \quad (1.18)$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, \omega) = \rho(\mathbf{r}, \omega) \quad (1.19)$$

These equations are consequences of (1.15) and (1.16), and the continuity equation (1.17) (*cf.* Section 1.1 on page 4). In fact, take the divergence of the Maxwell equations (1.15) and (1.16) and use (1.17), which gives $(\nabla \cdot (\nabla \times \mathbf{A}) = 0)$

$$i\omega\nabla \cdot \mathbf{B}(\mathbf{r}, \omega) = 0$$

$$i\omega\nabla \cdot \mathbf{D}(\mathbf{r}, \omega) = \nabla \cdot \mathbf{J}(\mathbf{r}, \omega) = i\omega\rho(\mathbf{r}, \omega)$$

Division by $i\omega$ (provided $\omega \neq 0$) then gives (1.18) and (1.19).

To summarize, in a source-free region the time-harmonic Maxwell equations are

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{r}, \omega) = ik_0(c_0\mathbf{B}(\mathbf{r}, \omega)) \\ \nabla \times (\eta_0\mathbf{H}(\mathbf{r}, \omega)) = -ik_0(c_0\eta_0\mathbf{D}(\mathbf{r}, \omega)) \end{cases} \quad (1.20)$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ is the intrinsic wave impedance of vacuum, $c_0 = 1/\sqrt{\epsilon_0\mu_0}$ the speed of light in vacuum, and $k_0 = \omega/c_0$ is the wave number¹¹ in vacuum. In equation (1.20) all field quantities in parenthesis have the same units, *i.e.*, that of the electric field. This form is the standard form of the Maxwell equations that we use in this textbook.

1.2.2 Constitutive relations

Constitutive relations model the interaction of the electromagnetic fields with materials. This topic is complex and well beyond the scope of this treatment. We limit ourselves to simple homogeneous, isotropic materials, which is the most simple material model. This model relates the electric and magnetic flux densities to the corresponding fields, *i.e.*,

$$\begin{cases} \mathbf{D}(\mathbf{r}, \omega) = \epsilon_0\epsilon(\omega)\mathbf{E}(\mathbf{r}, \omega) \\ \mathbf{B}(\mathbf{r}, \omega) = \mu_0\mu(\omega)\mathbf{H}(\mathbf{r}, \omega) \end{cases} \quad (1.21)$$

The parameters $\epsilon(\omega)$ and $\mu(\omega)$ are the (relative) permittivity and permeability of the medium, respectively. The isotropic model is used frequently and is a good model for many insulation materials, *e.g.*, glass, china, and many plastic materials.

¹¹More correctly, k_0 is the angular wave number in vacuum, and f/c_0 is the wave number in vacuum.

1.2.3 Poynting's theorem

In Section 1.1 we derived Poynting's theorem in vacuum, see (1.10) on page 5.

$$\nabla \cdot \{\mathbf{E}(t) \times \mathbf{H}(t)\} + \frac{1}{2} \frac{\partial}{\partial t} [\mu_0 |\mathbf{H}(t)|^2 + \epsilon_0 |\mathbf{E}(t)|^2] + \mathbf{E}(t) \cdot \mathbf{J}(t) = 0$$

The equation describes conservation of power and contains products of two fields. For a product of time harmonic fields, the most pertinent quantity is the time average over one period.¹² We denote the time average as $\langle \cdot \rangle$ and for Poynting's theorem we obtain

$$\langle \nabla \cdot \{\mathbf{E}(t) \times \mathbf{H}(t)\} \rangle + \frac{\mu_0}{2} \left\langle \frac{\partial}{\partial t} |\mathbf{H}(t)|^2 \right\rangle + \frac{\epsilon_0}{2} \left\langle \frac{\partial}{\partial t} |\mathbf{E}(t)|^2 \right\rangle + \langle \mathbf{E}(t) \cdot \mathbf{J}(t) \rangle = 0$$

The different terms in this quantity after a time average are

$$\langle \mathbf{E}(t) \times \mathbf{H}(t) \rangle = \frac{1}{2} \operatorname{Re} \{\mathbf{E}(\omega) \times \mathbf{H}^*(\omega)\} \quad (1.22)$$

and

$$\begin{aligned} \left\langle \frac{\partial}{\partial t} |\mathbf{H}(t)|^2 \right\rangle &= \frac{1}{2} \operatorname{Re} \{-i\omega |\mathbf{H}(\omega)|^2\} = 0 \\ \left\langle \frac{\partial}{\partial t} |\mathbf{E}(t)|^2 \right\rangle &= \frac{1}{2} \operatorname{Re} \{-i\omega |\mathbf{E}(\omega)|^2\} = 0 \\ \langle \mathbf{E}(t) \cdot \mathbf{J}(t) \rangle &= \frac{1}{2} \operatorname{Re} \{\mathbf{E}(\omega) \cdot \mathbf{J}^*(\omega)\} \end{aligned}$$

Poynting's theorem (balance of power) for time harmonic fields in vacuum, averaged over a period, becomes (Notice that the time average and the differentiation w.r.t. space commute, *i.e.*, $\langle \nabla \cdot \{\mathbf{E}(t) \times \mathbf{H}(t)\} \rangle = \nabla \cdot \langle \{\mathbf{E}(t) \times \mathbf{H}(t)\} \rangle$):

$$\operatorname{Re} \nabla \cdot \{\mathbf{E}(\omega) \times \mathbf{H}^*(\omega)\} + \operatorname{Re} \{\mathbf{E}(\omega) \cdot \mathbf{J}^*(\omega)\} = 0 \quad (1.23)$$

Of special interest is the case without currents $\mathbf{J} = \mathbf{0}$. Poynting's theorem is then simplified to

$$\operatorname{Re} \nabla \cdot \{\mathbf{E}(\omega) \times \mathbf{H}^*(\omega)\} = 0$$

Integration over a finite volume V with bounding surface S and outward pointing normal $\hat{\nu}$ then shows

$$\operatorname{Re} \iint_S \{\mathbf{E}(\omega) \times \mathbf{H}^*(\omega)\} \cdot \hat{\nu} \, dS = \operatorname{Re} \iiint_V \nabla \cdot \{\mathbf{E}(\omega) \times \mathbf{H}^*(\omega)\} \, dv = 0$$

¹²The time average of a product of two time harmonic fields $f_1(t)$ and $f_2(t)$ is easily obtained by an average over one period $T = 2\pi/\omega$.

$$\begin{aligned} \langle f_1(t)f_2(t) \rangle &= \frac{1}{T} \int_0^T f_1(t)f_2(t) \, dt = \frac{1}{T} \int_0^T \operatorname{Re} \{f_1(\omega)e^{-i\omega t}\} \operatorname{Re} \{f_2(\omega)e^{-i\omega t}\} \, dt \\ &= \frac{1}{4T} \int_0^T \{f_1(\omega)f_2(\omega)e^{-2i\omega t} + f_1^*(\omega)f_2^*(\omega)e^{2i\omega t} + f_1(\omega)f_2^*(\omega) + f_1^*(\omega)f_2(\omega)\} \, dt \\ &= \frac{1}{4} \{f_1(\omega)f_2^*(\omega) + f_1^*(\omega)f_2(\omega)\} = \frac{1}{2} \operatorname{Re} \{f_1(\omega)f_2^*(\omega)\} \end{aligned}$$

Here we have used the divergence theorem. This identity shows that the net flux of power through the surface S is zero.

1.2.4 Reciprocity

In this section we introduce the concept of reciprocity, which compares solutions to the Maxwell equations. As with the Poynting's theorem, we restrict ourselves to a vacuous region V . The first set of solution, denoted by the superscript a, *i.e.*, the fields are \mathbf{E}^a and \mathbf{H}^a . The second set of solution, which we denote by the superscript b, has fields \mathbf{E}^b and \mathbf{H}^b . All sources are assumed to be located outside the finite volume V .

We investigate the following surface integral over the bounding surface S :

$$\iint_S (\mathbf{E}^a \times \mathbf{H}^b - \mathbf{E}^b \times \mathbf{H}^a) \cdot \hat{\nu} \, dS = \iiint_V \nabla \cdot (\mathbf{E}^a \times \mathbf{H}^b - \mathbf{E}^b \times \mathbf{H}^a) \, dv$$

where we used the divergence theorem to transform the surface integral into a volume integral over V .

Now use the Maxwell equation $\nabla \times \mathbf{H} = -i\omega\epsilon_0\mathbf{E}$ and $\nabla \times \mathbf{E} = i\omega\mu_0\mathbf{H}$ and the differentiation rule $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = (\nabla \times \mathbf{a}) \cdot \mathbf{b} - \mathbf{a} \cdot (\nabla \times \mathbf{b})$ to rewrite the volume integral as

$$\begin{aligned} & \iint_S (\mathbf{E}^a \times \mathbf{H}^b - \mathbf{E}^b \times \mathbf{H}^a) \cdot \hat{\nu} \, dS \\ &= \iiint_V ((\nabla \times \mathbf{E}^a) \cdot \mathbf{H}^b - \mathbf{E}^a \cdot (\nabla \times \mathbf{H}^b) - (\nabla \times \mathbf{E}^b) \cdot \mathbf{H}^a + \mathbf{E}^b \cdot (\nabla \times \mathbf{H}^a)) \, dv \\ &= i\omega \iiint_V \{ \mu_0 \mathbf{H}^a \cdot \mathbf{H}^b + \epsilon_0 \mathbf{E}^a \cdot \mathbf{E}^b - \mu_0 \mathbf{H}^b \cdot \mathbf{H}^a - \epsilon_0 \mathbf{E}^b \cdot \mathbf{E}^a \} \, dv = 0 \end{aligned}$$

The conclusion therefore is that in a vacuous volume bounded by a surface S , two solutions always satisfy

$$\iint_S (\mathbf{E}^a \times \mathbf{H}^b - \mathbf{E}^b \times \mathbf{H}^a) \cdot \hat{\nu} \, dS = 0$$

This is Lorentz' reciprocity theorem.

1.2.5 Ellipse of polarization

A time harmonic field can be described in geometrical terms. All time harmonic fields oscillate in a fixed plane and the field follows the trace of an ellipse in this plane. The presentation in this section is coordinate-free, which is advantageous since the analysis can be made without referring to any specific coordinate system.

We consider the time harmonic field $\mathbf{E}(t)$ (all dependence on the space coordinates \mathbf{r} is suppressed in this section) at a fixed point in space. The time dependence of the field is

$$\mathbf{E}(t) = \text{Re} \{ \mathbf{E}_0 e^{-i\omega t} \} \quad (1.24)$$

where \mathbf{E}_0 is a constant complex vector (can depend on, *e.g.*, ω and \mathbf{r}), which Cartesian components are

$$\mathbf{E}_0 = \hat{\mathbf{x}}E_{0x} + \hat{\mathbf{y}}E_{0y} + \hat{\mathbf{z}}E_{0z} = \hat{\mathbf{x}}|E_{0x}|e^{i\alpha} + \hat{\mathbf{y}}|E_{0y}|e^{i\beta} + \hat{\mathbf{z}}|E_{0z}|e^{i\gamma}$$

and α , β and γ are the phase of the components, respectively.

First we observe that the vector $\mathbf{E}(t)$ in (1.24) for all times lies in a fixed plane in space. To see this, we express the complex vector \mathbf{E}_0 in its real and imaginary parts, \mathbf{E}_{0r} and \mathbf{E}_{0i} , respectively.

$$\mathbf{E}_0 = \mathbf{E}_{0r} + i\mathbf{E}_{0i}$$

The real vectors \mathbf{E}_{0r} and \mathbf{E}_{0i} are fixed in time, and their explicit Cartesian components are

$$\begin{aligned} \mathbf{E}_{0r} &= \hat{\mathbf{x}}|E_{0x}| \cos \alpha + \hat{\mathbf{y}}|E_{0y}| \cos \beta + \hat{\mathbf{z}}|E_{0z}| \cos \gamma \\ \mathbf{E}_{0i} &= \hat{\mathbf{x}}|E_{0x}| \sin \alpha + \hat{\mathbf{y}}|E_{0y}| \sin \beta + \hat{\mathbf{z}}|E_{0z}| \sin \gamma \end{aligned}$$

The vector $\mathbf{E}(t)$ in (1.24) is now rewritten as

$$\mathbf{E}(t) = \text{Re} \{ (\mathbf{E}_{0r} + i\mathbf{E}_{0i}) e^{-i\omega t} \} = \mathbf{E}_{0r} \cos \omega t + \mathbf{E}_{0i} \sin \omega t \quad (1.25)$$

from which we conclude that the vector $\mathbf{E}(t)$ lies in the plane spanned by the real vectors \mathbf{E}_{0r} and \mathbf{E}_{0i} for all times t . The normal to this plane is

$$\hat{\boldsymbol{\nu}} = \pm \frac{\mathbf{E}_{0r} \times \mathbf{E}_{0i}}{|\mathbf{E}_{0r} \times \mathbf{E}_{0i}|}$$

provided that $\mathbf{E}_{0r} \times \mathbf{E}_{0i} \neq \mathbf{0}$. In the case $\mathbf{E}_{0r} \times \mathbf{E}_{0i} = \mathbf{0}$, *i.e.*, the two real vectors \mathbf{E}_{0r} and \mathbf{E}_{0i} are parallel, the field \mathbf{E} oscillates along a fixed line in space, and no plane can be defined.

In general, the real vectors \mathbf{E}_{0r} and \mathbf{E}_{0i} , which span the plane in which the vector $\mathbf{E}(t)$ oscillates, are not orthogonal. However, it is convenient to use orthogonal vectors. To this end, we introduce two new orthogonal vectors, \mathbf{a} and \mathbf{b} , which are linear combinations of the vectors \mathbf{E}_{0r} and \mathbf{E}_{0i} . Let

$$\begin{cases} \mathbf{a} = \mathbf{E}_{0r} \cos \vartheta + \mathbf{E}_{0i} \sin \vartheta \\ \mathbf{b} = -\mathbf{E}_{0r} \sin \vartheta + \mathbf{E}_{0i} \cos \vartheta \end{cases} \quad (1.26)$$

where the angle $\vartheta \in [-\pi/4, \pi/4]$ is defined as

$$\tan 2\vartheta = \frac{2\mathbf{E}_{0r} \cdot \mathbf{E}_{0i}}{|\mathbf{E}_{0r}|^2 - |\mathbf{E}_{0i}|^2}$$

By this construction \mathbf{a} and \mathbf{b} are orthogonal, since

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (\mathbf{E}_{0r} \cos \vartheta + \mathbf{E}_{0i} \sin \vartheta) \cdot (-\mathbf{E}_{0r} \sin \vartheta + \mathbf{E}_{0i} \cos \vartheta) \\ &= -(|\mathbf{E}_{0r}|^2 - |\mathbf{E}_{0i}|^2) \sin \vartheta \cos \vartheta + \mathbf{E}_{0r} \cdot \mathbf{E}_{0i} (\cos^2 \vartheta - \sin^2 \vartheta) \\ &= -\frac{1}{2} (|\mathbf{E}_{0r}|^2 - |\mathbf{E}_{0i}|^2) \sin 2\vartheta + \mathbf{E}_{0r} \cdot \mathbf{E}_{0i} \cos 2\vartheta = 0\end{aligned}$$

by the definition of the angle ϑ .

The vectors \mathbf{E}_{0r} and \mathbf{E}_{0i} can be expressed in the vectors \mathbf{a} and \mathbf{b} . The result is

$$\begin{cases} \mathbf{E}_{0r} = \mathbf{a} \cos \vartheta - \mathbf{b} \sin \vartheta \\ \mathbf{E}_{0i} = \mathbf{a} \sin \vartheta + \mathbf{b} \cos \vartheta \end{cases}$$

i.e.,

$$\mathbf{E}_0 = \mathbf{E}_{0r} + i\mathbf{E}_{0i} = (\mathbf{a} \cos \vartheta - \mathbf{b} \sin \vartheta) + i(\mathbf{a} \sin \vartheta + \mathbf{b} \cos \vartheta) = e^{i\vartheta}(\mathbf{a} + i\mathbf{b}) \quad (1.27)$$

This representation also implies a simple form of the magnitude of the complex vector \mathbf{E}_0 , *i.e.*,

$$|\mathbf{E}_0|^2 = \mathbf{E}_0 \cdot \mathbf{E}_0^* = (\mathbf{a} + i\mathbf{b}) \cdot (\mathbf{a} - i\mathbf{b}) = a^2 + b^2$$

Inserting in (1.25) we get the physical field, *i.e.*,

$$\begin{aligned}\mathbf{E}(t) &= \mathbf{E}_{0r} \cos \omega t + \mathbf{E}_{0i} \sin \omega t \\ &= (\mathbf{a} \cos \vartheta - \mathbf{b} \sin \vartheta) \cos \omega t + (\mathbf{a} \sin \vartheta + \mathbf{b} \cos \vartheta) \sin \omega t \\ &= \mathbf{a} \cos(\omega t - \vartheta) + \mathbf{b} \sin(\omega t - \vartheta)\end{aligned} \quad (1.28)$$

The vectors \mathbf{a} and \mathbf{b} can be used as a basis in an orthogonal coordinate system in the plane where the field \mathbf{E} oscillates. From a comparison with the equation of the ellipse in the xy -planet (half axes a and b along the x - and the y -axes, respectively)

$$\begin{cases} x = a \cos \phi \\ y = b \sin \phi \end{cases}$$

and (1.28), we conclude that the field \mathbf{E} traces an ellipse in the plane spanned by the vectors \mathbf{a} and \mathbf{b} and that these vectors are the half axes (both the direction and size) of the ellipse, see Figure 1.4. From (1.28) we also see that the field \mathbf{E} is directed along the half axis \mathbf{a} when $\omega t = \vartheta + 2n\pi$, and that the field \mathbf{E} is directed along the other half axis \mathbf{b} when $\omega t = \vartheta + \pi/2 + 2n\pi$. The angle ϑ is the parameter that marks where on the ellipse the field \mathbf{E} is directed at $t = 0$, *i.e.*,

$$\mathbf{E}(t = 0) = \mathbf{a} \cos \vartheta - \mathbf{b} \sin \vartheta$$

and the vector \mathbf{E} moves along the ellipse in a direction from \mathbf{a} to \mathbf{b} (shortest way). The vectors \mathbf{a} and \mathbf{b} describes the polarization state¹³ of the field \mathbf{E} completely, except for the phase angle ϑ .

¹³Do not mix the concept of polarization of the material, \mathbf{P} , with the polarization of a vector field.

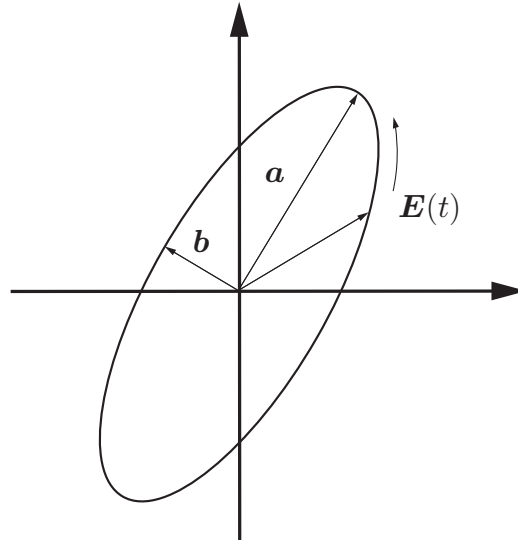


Figure 1.4: The ellipse of polarization and its half axes \mathbf{a} and \mathbf{b} .

$i\hat{\mathbf{e}} \cdot (\mathbf{E}_0 \times \mathbf{E}_0^*)$	Polarization
$= 0$	Linear polarization
> 0	Right-handed elliptic polarization
< 0	Left-handed elliptic polarization

Table 1.3: Table of the state of polarization of a time harmonic field.

We are now classifying the polarization state of the time harmonic field $\mathbf{E}(t)$. This field can either be rotating along the elliptic curve in a clockwise or a counterclockwise direction. Without a preferred direction in space, the direction of rotation is a relative concept — depending on which side of the plane we observe the oscillations. From the direction of the power flow of the electromagnetic field at the point of observation, $\langle \mathbf{S}(t) \rangle$, we define a preferred direction in space. Let $\hat{\mathbf{e}}$ be the normal to the plane of polarization, such that $\langle \mathbf{S}(t) \rangle \cdot \hat{\mathbf{e}} > 0$. We use this unit vector $\hat{\mathbf{e}}$ as a reference direction.

The polarization of the field is now classified according to the sign of the component of $i\mathbf{E}_0 \times \mathbf{E}_0^* = 2\mathbf{E}_{0r} \times \mathbf{E}_{0i} = 2\mathbf{a} \times \mathbf{b}$ on $\hat{\mathbf{e}}$, see Table 1.3. The field vector either rotates counterclockwise (right-handed elliptic polarization) or clockwise (left-handed elliptic polarization) in the \mathbf{a} - \mathbf{b} -plane, see Figure 1.5, if we assume that the unit vector $\hat{\mathbf{e}}$ is directed **towards** the observer, and that the vectors \mathbf{a} and \mathbf{b} have the position depicted in Figure 1.4.¹⁴

The degenerated case, when the vectors \mathbf{E}_{0r} and \mathbf{E}_{0i} are parallel, implies that the field vector moves along a line through the origin — therefore the notion linear

¹⁴In the literature there are also occur the opposite definition of right- and left-handed elliptic polarization. Examples with the opposite definition are: [5], [14], and [15]. In this book, we are using the same definition as, *e.g.*, [2], [4], [6], and [7]. Our definition also coincides with the IEEE-standard.

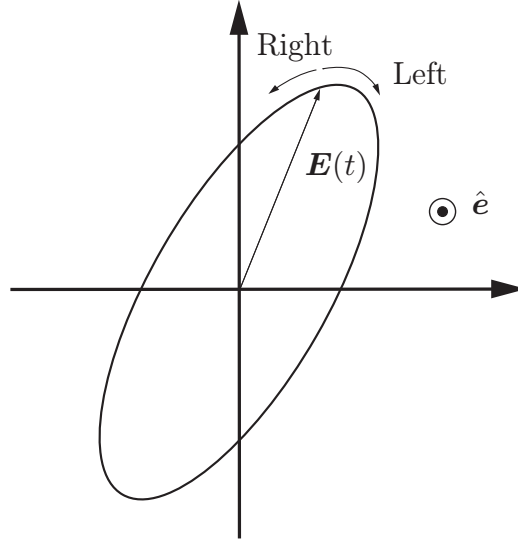


Figure 1.5: The ellipse of polarization and the definition of right- and left-handed polarization. The unit vector $\hat{\mathbf{e}}$ perpendicular to the plane in which the field vector $\mathbf{E}(t)$ oscillates and satisfies $\langle \mathbf{S}(t) \rangle \cdot \hat{\mathbf{e}} > 0$. It is assumed that the vectors \mathbf{a} and \mathbf{b} have the position depicted in Figure 1.4.

polarization. The linear polarization is characterized by $\mathbf{E}_0 \times \mathbf{E}_0^* = \mathbf{0}$. The case of a linear polarization can be viewed as a special case of an elliptic polarization, where one of the half axes is zero.

One special case of elliptic polarization is particularly important. This occurs when the half axes of the ellipse, \mathbf{a} and \mathbf{b} , have the same length, and the ellipse is a circle. We then have circular polarization. Whether the polarization is circular or not is decided by testing if $\mathbf{E}_0 \cdot \mathbf{E}_0 = 0$. To see this, we use (1.27) and the orthogonality between the vectors \mathbf{a} and \mathbf{b} , and we get

$$\mathbf{E}_0 \cdot \mathbf{E}_0 = e^{2i\vartheta} (\mathbf{a} + i\mathbf{b}) \cdot (\mathbf{a} + i\mathbf{b}) = e^{2i\vartheta} (|\mathbf{a}|^2 - |\mathbf{b}|^2)$$

The ellipse of polarization is therefore a circle, $a = |\mathbf{a}| = |\mathbf{b}| = b$, if and only if $\mathbf{E}_0 \cdot \mathbf{E}_0 = 0$. The direction of rotation is determined by the sign of the quantity $i\hat{\mathbf{e}} \cdot (\mathbf{E}_0 \times \mathbf{E}_0^*)$. Right (left) circular polarization is abbreviated RCP (LCP).

Another, more convenient, way of determining whether the polarization is circular or not is to study the quantity, use (1.27)

$$i\hat{\mathbf{e}} \cdot (\hat{\mathbf{p}}_e \times \hat{\mathbf{p}}_e^*) = \frac{2\hat{\mathbf{e}} \cdot (\mathbf{a} \times \mathbf{b})}{|\mathbf{a}|^2 + |\mathbf{b}|^2} = \frac{\pm 2ab}{a^2 + b^2} = \pm \left(1 - \frac{(a-b)^2}{a^2 + b^2} \right) \quad (1.29)$$

where $\hat{\mathbf{p}}_e = \mathbf{E}_0/|\mathbf{E}_0|$. If this quantity is ± 1 , we have RCP (upper sign) or LCP (lower sign). It is therefore convenient to define a polarization state quantity Π as

$$\Pi = i\hat{\mathbf{e}} \cdot (\hat{\mathbf{p}}_e \times \hat{\mathbf{p}}_e^*) \quad (1.30)$$

Π	Polarization	χ	$\hat{\mathbf{e}} \cdot (\mathbf{a} \times \mathbf{b})$
-1	Circular polarization LCP	$\pi/4$	-1
< 0	Left-handed elliptic polarization	$(0, \pi/2)$	-1
0	Linear polarization LP	$0, \pi/2$	± 1
> 0	Right-handed elliptic polarization	$(0, \pi/2)$	1
1	Circular polarization RCP	$\pi/4$	1

Table 1.4: Table of the polarization state Π of a time harmonic field.

This quantity is always in the interval $[-1, 1]$. $\Pi = -1$ corresponds to LCP, $\Pi = 0$ corresponds to LP, and $\Pi = 1$ corresponds to RCP. We summarize these observations in Table 1.4.

In terms of the notation above, a general polarization state is given by

$$\hat{\mathbf{p}}_e = e^{i\vartheta} \frac{(\mathbf{a} + i\mathbf{b})}{\sqrt{a^2 + b^2}}$$

Define the angle χ by

$$\tan \chi = \frac{b}{a}, \quad \chi \in [0, \pi/2] \quad \Rightarrow \quad \begin{cases} \cos \chi = \frac{a}{\sqrt{a^2 + b^2}} \\ \sin \chi = \frac{b}{\sqrt{a^2 + b^2}} \end{cases}$$

and in terms of the natural orthonormal basis $\{\hat{\mathbf{a}}, \hat{\mathbf{b}}\}$ aligned along the two half axis of the polarization ellipse, we get

$$\hat{\mathbf{p}}_e = e^{i\vartheta} \frac{(\mathbf{a} + i\mathbf{b})}{\sqrt{a^2 + b^2}} = e^{i\vartheta} (\hat{\mathbf{a}} \cos \chi + i\hat{\mathbf{b}} \sin \chi) \quad (1.31)$$

and also the polarization state

$$\Pi = i\hat{\mathbf{e}} \cdot (\hat{\mathbf{p}}_e \times \hat{\mathbf{p}}_e^*) = 2\hat{\mathbf{e}} \cdot (\hat{\mathbf{a}} \times \hat{\mathbf{b}}) \sin \chi \cos \chi = \hat{\mathbf{e}} \cdot (\hat{\mathbf{a}} \times \hat{\mathbf{b}}) \sin 2\chi$$

The canonical form of a RCP field is

$$\mathbf{E}_0 = E_0 (\hat{\mathbf{a}} + i\hat{\mathbf{b}})$$

and the canonical form of a LCP field is

$$\mathbf{E}_0 = E_0 (\hat{\mathbf{a}} - i\hat{\mathbf{b}})$$

if $\{\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{e}}\}$ forms a right-handed orthonormal basis.

Example 1.1

The most general harmonic field in the $\hat{\mathbf{e}}_1$ - $\hat{\mathbf{e}}_2$ -plane has the form (we assume $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}\}$ forms a right-handed orthonormal basis)

$$\mathbf{E}(t) = \hat{\mathbf{e}}_1 A \cos(\omega t - \alpha) + \hat{\mathbf{e}}_2 B \cos(\omega t - \beta)$$

where $A \geq 0$, $B \geq 0$, and α and β are real angles. The corresponding complex vector \mathbf{E}_0 is

$$\begin{cases} \mathbf{E}(t) = \operatorname{Re} \{ \mathbf{E}_0 e^{-i\omega t} \} \\ \mathbf{E}_0 = A e^{i\alpha} \hat{\mathbf{e}}_1 + B e^{i\beta} \hat{\mathbf{e}}_2 \end{cases} \quad (1.32)$$

which implies

$$i\mathbf{E}_0 \times \mathbf{E}_0^* = iABe^{i(\alpha-\beta)} \hat{\mathbf{e}}_3 - iABe^{-i(\alpha-\beta)} \hat{\mathbf{e}}_3 = -2AB\hat{\mathbf{e}}_3 \sin(\alpha - \beta)$$

From this we conclude that the field is

$$\begin{aligned} &\text{Left-handed polarization if } 0 < \alpha - \beta < \pi \\ &\text{Right-handed polarization if } \pi < \alpha - \beta < 2\pi \\ &\text{Linear polarization if } \alpha = \beta \text{ or } \alpha = \beta + \pi \end{aligned}$$

where the inequalities are interpreted mod 2π .

The real and imaginary part of \mathbf{E}_0 are

$$\begin{cases} \mathbf{E}_{0r} = \hat{\mathbf{e}}_1 A \cos \alpha + \hat{\mathbf{e}}_2 B \cos \beta \\ \mathbf{E}_{0i} = \hat{\mathbf{e}}_1 A \sin \alpha + \hat{\mathbf{e}}_2 B \sin \beta \end{cases}$$

From equation (1.26) we have

$$\begin{cases} \mathbf{a} = (\hat{\mathbf{e}}_1 A \cos \alpha + \hat{\mathbf{e}}_2 B \cos \beta) \cos \vartheta + (\hat{\mathbf{e}}_1 A \sin \alpha + \hat{\mathbf{e}}_2 B \sin \beta) \sin \vartheta \\ \mathbf{b} = -(\hat{\mathbf{e}}_1 A \cos \alpha + \hat{\mathbf{e}}_2 B \cos \beta) \sin \vartheta + (\hat{\mathbf{e}}_1 A \sin \alpha + \hat{\mathbf{e}}_2 B \sin \beta) \cos \vartheta \end{cases}$$

where the angle ϑ is determined by

$$\tan 2\vartheta = \frac{A^2 \sin 2\alpha + B^2 \sin 2\beta}{A^2 \cos 2\alpha + B^2 \cos 2\beta} \quad (1.33)$$

which implies that the half axes of the ellipse are

$$\begin{cases} \mathbf{a} = A\hat{\mathbf{e}}_1 \cos(\vartheta - \alpha) + B\hat{\mathbf{e}}_2 \cos(\vartheta - \beta) \\ \mathbf{b} = -A\hat{\mathbf{e}}_1 \sin(\vartheta - \alpha) - B\hat{\mathbf{e}}_2 \sin(\vartheta - \beta) \end{cases}$$

The length of the half axes are

$$\begin{cases} a = \sqrt{A^2 \cos^2(\vartheta - \alpha) + B^2 \cos^2(\vartheta - \beta)} \\ b = \sqrt{A^2 \sin^2(\vartheta - \alpha) + B^2 \sin^2(\vartheta - \beta)} \end{cases} \quad (1.34)$$

and the angles ϕ_a and ϕ_b between the $\hat{\mathbf{e}}_1$ -axis and the half axes \mathbf{a} and \mathbf{b} are determined by

$$\begin{cases} \tan \phi_a = \frac{B \cos(\vartheta - \beta)}{A \cos(\vartheta - \alpha)} \\ \tan(\phi_b - \pi) = \frac{B \sin(\vartheta - \beta)}{A \sin(\vartheta - \alpha)} \end{cases} \quad (1.35)$$

respectively. ■

Example 1.2

Construct the harmonic field, oscillating in the $\hat{\mathbf{e}}_1$ - $\hat{\mathbf{e}}_2$ -plane, satisfying the following specification (see also Figure 1.6):

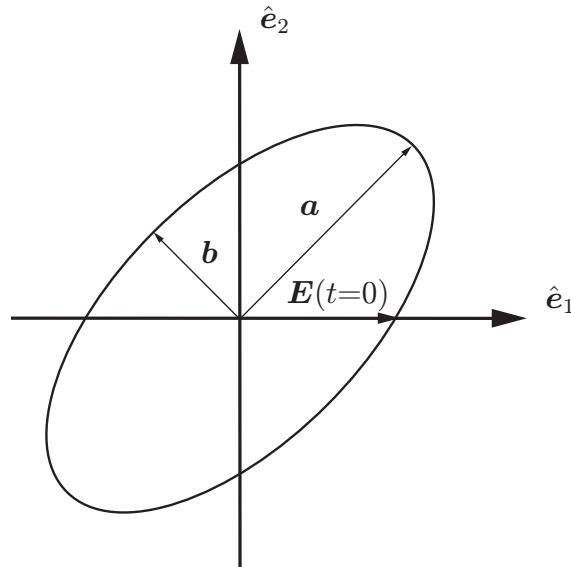


Figure 1.6: Ellipse of polarization in Example 1.2.

- The field is at time $t = 0$ polarized along the \hat{e}_1 -axis and strength E (a given real constant), *i.e.*, $\mathbf{E}(t = 0) = \hat{e}_1 E$.
- The quotient between the axes of the ellipse is $\varepsilon = b/a$. The axis \mathbf{a} , with the length a , is located in the first quadrant, and the angle between \mathbf{a} and the \hat{e}_1 -axis is ϕ .
- The field has right-handed elliptic polarization ($\langle \mathbf{S}(t) \rangle$ is assumed to be directed along $\hat{e}_3 = \hat{e}_1 \times \hat{e}_2$).

We assume $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ forms a right-handed orthonormal basis. Determine the real constants E_1 , E_2 , α , and β in the expression

$$\mathbf{E}(t) = \hat{e}_1 E_1 \cos(\omega t - \alpha) + \hat{e}_2 E_2 \cos(\omega t - \beta)$$

i.e., determine the amplitude and the phase of the \hat{e}_1 - and \hat{e}_2 -components.

Solution: Introduce the half axes of the ellipse.

$$\begin{cases} \mathbf{a} = \frac{a}{\sqrt{1 + \tan^2 \phi}} (\hat{e}_1 + \hat{e}_2 \tan \phi) \\ \mathbf{b} = \frac{a\varepsilon}{\sqrt{1 + \tan^2 \phi}} (-\hat{e}_1 \tan \phi + \hat{e}_2) \end{cases}$$

which implies that the length of the vectors \mathbf{a} and \mathbf{b} are a and b , respectively, and, moreover, this choice gives a right-handed elliptic polarization of the field since

$$(\mathbf{a} \times \mathbf{b}) \cdot \hat{e}_3 = a^2 \varepsilon > 0$$

Now determine the angle ϑ in the expression $\mathbf{E}_0 = e^{i\vartheta}(\mathbf{a} + i\mathbf{b})$.

$$\mathbf{E}(t) = \mathbf{E}_{0r} \cos \omega t + \mathbf{E}_{0i} \sin \omega t = \mathbf{a} \cos(\omega t - \vartheta) + \mathbf{b} \sin(\omega t - \vartheta)$$

At time $t = 0$ we have

$$\mathbf{E}(0) = \mathbf{E}_{0r} = \mathbf{a} \cos \vartheta - \mathbf{b} \sin \vartheta = E \hat{e}_1$$

i.e., the components satisfy

$$\begin{cases} \frac{a}{\sqrt{1 + \tan^2 \phi}} (\cos \vartheta + \varepsilon \tan \phi \sin \vartheta) = E \\ \frac{a}{\sqrt{1 + \tan^2 \phi}} (\tan \phi \cos \vartheta - \varepsilon \sin \vartheta) = 0 \end{cases}$$

with solution

$$\begin{cases} \frac{a}{\sqrt{1 + \tan^2 \phi}} \cos \vartheta = \frac{E}{1 + \tan^2 \phi} \\ \frac{a}{\sqrt{1 + \tan^2 \phi}} \sin \vartheta = \frac{E \tan \phi}{\varepsilon (1 + \tan^2 \phi)} \end{cases}$$

and we have

$$\frac{a}{\sqrt{1 + \tan^2 \phi}} e^{i\vartheta} = \frac{E (\varepsilon + i \tan \phi)}{\varepsilon (1 + \tan^2 \phi)}$$

We get

$$\begin{aligned} \mathbf{E}_0 = e^{i\vartheta} (\mathbf{a} + i\mathbf{b}) &= \frac{E (\varepsilon + i \tan \phi)}{\varepsilon (1 + \tan^2 \phi)} \{ (\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \tan \phi) + i\varepsilon (-\hat{\mathbf{e}}_1 \tan \phi + \hat{\mathbf{e}}_2) \} \\ &= E \left\{ \hat{\mathbf{e}}_1 \left(1 + i \frac{\tan \phi}{1 + \tan^2 \phi} \frac{1 - \varepsilon^2}{\varepsilon} \right) + i\hat{\mathbf{e}}_2 \frac{\varepsilon^2 + \tan^2 \phi}{\varepsilon (1 + \tan^2 \phi)} \right\} \\ &= E \left\{ \hat{\mathbf{e}}_1 \left(1 + i \frac{1 - \varepsilon^2}{2\varepsilon} \sin 2\phi \right) + i\hat{\mathbf{e}}_2 \left(\varepsilon \cos^2 \phi + \frac{1}{\varepsilon} \sin^2 \phi \right) \right\} \end{aligned}$$

From this expression of \mathbf{E}_0 we can identify the amplitudes A and B and the phases α and β in

$$\mathbf{E}(t) = \hat{\mathbf{e}}_1 A \cos(\omega t - \alpha) + \hat{\mathbf{e}}_2 B \cos(\omega t - \beta)$$

by rewriting the complex vector \mathbf{E}_0 in polar form

$$\mathbf{E}_0 = \hat{\mathbf{e}}_1 A e^{i\alpha} + \hat{\mathbf{e}}_2 B e^{i\beta}$$

i.e.,

$$\begin{cases} A = \sqrt{1 + \frac{(1 - \varepsilon^2)^2}{4\varepsilon^2} \sin^2 2\phi} \\ B = \varepsilon \cos^2 \phi + \frac{1}{\varepsilon} \sin^2 \phi \end{cases} \quad \begin{cases} \alpha = \arctan \left(\frac{1 - \varepsilon^2}{2\varepsilon} \sin 2\phi \right) \\ \beta = \frac{\pi}{2} \end{cases}$$

■

1.3 Green's functions and dyadics

To analyze the electromagnetic fields we need some results from Green's function techniques and the scalar and vector potentials.

1.3.1 Green's functions in isotropic media

In this section, we quantify the electromagnetic field in a homogeneous, isotropic material due to a given current density \mathbf{J} , which quantifies all sources everywhere in space. The Maxwell equations are then, see (1.15) and (1.16)

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{r}, \omega) = i\omega \mathbf{B}(\mathbf{r}, \omega) \\ \nabla \times \mathbf{H}(\mathbf{r}, \omega) = \mathbf{J}(\mathbf{r}, \omega) - i\omega \mathbf{D}(\mathbf{r}, \omega) \end{cases} \quad \mathbf{r} \in \mathbb{R}^3$$

Our goal here is to solve these equations for a given current density $\mathbf{J}(\mathbf{r}, \omega)$, which is assumed to vanish outside a finite volume V . We show that this is possible provided we can determine the Green's function of the problem.

The assumption of a homogeneous, isotropic material implies that the constitutive relations are

$$\begin{cases} \mathbf{D}(\mathbf{r}, \omega) = \epsilon_0 \epsilon(\omega) \mathbf{E}(\mathbf{r}, \omega) \\ \mathbf{B}(\mathbf{r}, \omega) = \mu_0 \mu(\omega) \mathbf{H}(\mathbf{r}, \omega) \end{cases} \quad (1.36)$$

where $\epsilon(\omega)$ and $\mu(\omega)$ are complex constants independent of \mathbf{r} , but they can still depend on the (angular) frequency ω . If we combine these equations, we get

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{r}, \omega) = ik\eta_0\eta(\omega)\mathbf{H}(\mathbf{r}, \omega) \\ \eta_0\eta(\omega)\nabla \times \mathbf{H}(\mathbf{r}, \omega) = \eta_0\eta(\omega)\mathbf{J}(\mathbf{r}, \omega) - ik\mathbf{E}(\mathbf{r}, \omega) \end{cases}$$

where the **wave number** k is

$$k = \omega (\epsilon_0 \mu_0 \epsilon(\omega) \mu(\omega))^{1/2} = \frac{\omega}{c_0} (\epsilon(\omega) \mu(\omega))^{1/2}$$

and the **wave impedance** of vacuum, η_0 , and the relative impedance, η , are defined as

$$\eta_0 = \left(\frac{\mu_0}{\epsilon_0} \right)^{1/2} \quad \eta(\omega) = \left(\frac{\mu(\omega)}{\epsilon(\omega)} \right)^{1/2}$$

Consequently, we get

$$\nabla \times (\nabla \times \mathbf{E}(\mathbf{r}, \omega)) = ik\eta_0\eta(\omega)\nabla \times \mathbf{H}(\mathbf{r}, \omega) = ik\eta_0\eta(\omega)\mathbf{J}(\mathbf{r}, \omega) + k^2\mathbf{E}(\mathbf{r}, \omega)$$

or

$$\boxed{\nabla \times (\nabla \times \mathbf{E}(\mathbf{r}, \omega)) - k^2\mathbf{E}(\mathbf{r}, \omega) = ik\eta_0\eta(\omega)\mathbf{J}(\mathbf{r}, \omega) = i\omega\mu_0\mu(\omega)\mathbf{J}(\mathbf{r}, \omega)} \quad (1.37)$$

1.3.2 Potentials and gauge transformations

Equation (1.37) is the partial differential equation that the electric field satisfies for a given current density distributions \mathbf{J} . The solution to this equation then determines the other fields \mathbf{D} , \mathbf{B} and \mathbf{H} by the Maxwell equations and the constitutive relations (1.36).

The partial differential equation (1.37) is a system of three coupled equations; one equation for each Cartesian component. The fact that it couples all components to each other makes it hard to solve the equation as it stands. We are now looking for an alternative way of writing this system of equations such that each Cartesian component decouples. One way of accomplishing this is to introduce the vector potential \mathbf{A} defined by

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (1.38)$$

The existence of a vector potential is guaranteed by $\nabla \cdot \mathbf{B} = 0$.

The scalar potential ϕ is defined by Faraday's law.

$$\nabla \times \mathbf{E} = i\omega \mathbf{B} = i\omega \nabla \times \mathbf{A}$$

We get

$$\nabla \times (\mathbf{E} - i\omega \mathbf{A}) = \mathbf{0}$$

which implies that there is a function ϕ such that¹⁵

$$\mathbf{E} - i\omega \mathbf{A} = -\nabla \phi$$

or

$$\mathbf{E} = i\omega \mathbf{A} - \nabla \phi \quad (1.39)$$

The magnetic flux density \mathbf{B} and the electric field \mathbf{E} can therefore be computed from the vector potential \mathbf{A} and the scalar potential ϕ .

The vector potential \mathbf{A} and the scalar potential ϕ are not uniquely defined. If \mathbf{A} and ϕ are replaced by

$$\begin{cases} \mathbf{A}' = \mathbf{A} + \nabla \psi \\ \phi' = \phi + i\omega \psi \end{cases}$$

where ψ is an arbitrary (differentiable) function, the magnetic flux density and the electric field remain the same since

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \nabla \times \nabla \psi = \nabla \times \mathbf{A}$$

and

$$i\omega \mathbf{A}' - \nabla \phi' = i\omega \mathbf{A} + i\omega \nabla \psi - \nabla \phi - i\omega \nabla \psi = i\omega \mathbf{A} - \nabla \phi$$

As a consequence, we obtain the same physical fields, \mathbf{E} and \mathbf{B} , from both the unprimed and primed potentials. This type of transformation is called a gauge transformation.¹⁶

The lack of uniqueness of the vector potential \mathbf{A} and the scalar potential ϕ is now used to find a more simple equation for the vector potential \mathbf{A} than equation

¹⁵To make the existence of ϕ clear, we assume the domain of investigation is a simply connected domain. This is the case for \mathbb{R}^3 .

¹⁶The name gauge often raises questions. The mathematician Hermann Weyl coined the name in 1921 when he presented a theory on charged particles. In the theory lengths could be scaled differently at different points in space. The name gauge, which has the meaning of scaling, has survived.

(1.37) for the electric field \mathbf{E} . Therefore, we let the vector potential \mathbf{A} and the scalar potential ϕ satisfy a constraint. Several possibilities arise, but the one that is most used is the Lorenz' condition¹⁷ or Lorenz' gauge.

$$\boxed{\nabla \cdot \mathbf{A} = \frac{ik^2}{\omega} \phi} \quad (1.40)$$

It is always possible to find \mathbf{A} and ϕ such that this condition is fulfilled. In fact, assume we have a vector potential \mathbf{A}_0 and a scalar potential ϕ_0 that satisfy

$$\begin{cases} \mathbf{B} = \nabla \times \mathbf{A}_0 \\ \mathbf{E} = i\omega \mathbf{A}_0 - \nabla \phi_0 \end{cases}$$

but not (1.40). Then define a new vector potential \mathbf{A} and a new scalar potential ϕ by

$$\begin{cases} \mathbf{A} = \mathbf{A}_0 - \nabla \psi \\ \phi = \phi_0 - i\omega \psi \end{cases}$$

where the function ψ is an arbitrary solution to the inhomogeneous Helmholtz' equation,^{18,19} *i.e.*,

$$\nabla^2 \psi + k^2 \psi = \nabla \cdot \mathbf{A}_0 - \frac{ik^2}{\omega} \phi_0$$

Note that the right-hand side of this equation is known and non-zero, since we assumed that \mathbf{A}_0 and ϕ_0 do not satisfy Lorenz' condition. The physical fields \mathbf{E} and \mathbf{B} are not affected by this change of vector potential and scalar potential since the change $\mathbf{A}_0 \rightarrow \mathbf{A}$ and $\phi_0 \rightarrow \phi$ is a gauge transformation.

The new potentials \mathbf{A} and ϕ satisfy Lorenz' condition, which the following computations show

$$\begin{aligned} \nabla \cdot \mathbf{A} - \frac{ik^2}{\omega} \phi &= \nabla \cdot (\mathbf{A}_0 - \nabla \psi) - \frac{ik^2}{\omega} (\phi_0 - i\omega \psi) \\ &= \nabla \cdot \mathbf{A}_0 - \frac{ik^2}{\omega} \phi_0 - \nabla^2 \psi - k^2 \psi = 0 \end{aligned}$$

by the definition of the function ψ . It is therefore always possible to choose a vector potential \mathbf{A} and scalar potential ϕ such that Lorenz' condition is satisfied.

Even if the vector potential \mathbf{A} and the scalar potential ϕ satisfy Lorenz' condition, they are still not uniquely defined. This becomes clear by a similar analysis as above, but with $\mathbf{A}_0 = \mathbf{0}$ and $\phi_0 = 0$. Lorenz' condition is then satisfied from

¹⁷Note that it was the Danish physicist Ludvig Valentin Lorenz who first formulated this condition in a paper from 1867. The Dutch physicist Hendrik Antoon Lorentz made a similar discovery, but later. Therefore, the condition should be credited to Lorenz' name, and not Lorentz. Additional details can be found in [16].

¹⁸Hermann Ludwig Ferdinand von Helmholtz (1821–1894), German physician and physicist.

¹⁹A solution to this equation always exist (in the vector case) as shown later in this section.

the start, but nothing prevents us from changing the potentials by choosing ψ that satisfies the homogeneous Helmholtz' equation

$$\nabla^2\psi + k^2\psi = 0$$

The partial differential equation that \mathbf{A} satisfies becomes very simple, if we let Lorenz' condition be satisfied. Insert (1.39) in (1.37).

$$\nabla \times (\nabla \times (i\omega\mathbf{A} - \nabla\phi)) - k^2(i\omega\mathbf{A} - \nabla\phi) = i\omega\mu_0\mu\mathbf{J}$$

Then use $\nabla \times \nabla\phi = \mathbf{0}$ and introduce Lorenz' condition (1.40). We get

$$\nabla \times (\nabla \times \mathbf{A}) - k^2\mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) = \mu_0\mu\mathbf{J}$$

Moreover, we have

$$\nabla^2\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) \quad (1.41)$$

which is easily seen by evaluating the identity in Cartesian components. Finally, the vector potential \mathbf{A} satisfies the following equation:

$$\boxed{\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu_0\mu\mathbf{J}} \quad (1.42)$$

Provided we can solve this equation, we can determine ϕ by the use of (1.40) and the fields \mathbf{E} and \mathbf{B} by (1.39) and (1.38), respectively. The fields \mathbf{D} and \mathbf{H} are then determined by the constitutive relations. In general, (1.42) is more easy to solve than (1.37), since the different Cartesian components do not couple to each other in (1.42).

1.3.3 Canonical problem in free space

The solution to (1.42) is obtained by first solving a canonical problem with a simple source term. Suppose we have a solution $g(k, \mathbf{r}, \mathbf{r}')$ to the following equation:

$$\nabla^2 g(k, \mathbf{r}, \mathbf{r}') + k^2 g(k, \mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'), \quad \mathbf{r}, \mathbf{r}' \in \mathbb{R}^3 \quad (1.43)$$

where the differentiations are made w.r.t. the variable \mathbf{r} and where $\delta(\mathbf{r} - \mathbf{r}')$ denotes the three-dimensional delta distribution or delta function, *i.e.*, its representation in the Cartesian coordinated system is

$$\delta(\mathbf{r} - \mathbf{r}') = \delta(x - x')\delta(y - y')\delta(z - z')$$

The function $g(k, \mathbf{r}, \mathbf{r}')$ is the Green's function²⁰ of the scalar Helmholtz' equation. The solution to our original problem (1.42) then is

$$\mathbf{A}(\mathbf{r}) = \mu_0\mu \iiint_V g(k, \mathbf{r}, \mathbf{r}')\mathbf{J}(\mathbf{r}') dv' \quad (1.44)$$

²⁰George Green (1793–1841), British mathematician and physicist.

where dv' denotes the volume element $dx' dy' dz'$ (integration is made over the variable \mathbf{r}') and the domain of integration is the entire space or effectively the region in space where $\mathbf{J} \neq \mathbf{0}$.

Formally, we see that (1.44) is a solution to (1.42) by inserting the solution and interchanging the order of differentiation and integration. We have

$$\begin{aligned} \nabla^2 \mathbf{A}(\mathbf{r}) + k^2 \mathbf{A}(\mathbf{r}) &= \mu_0 \mu \iiint_V (\nabla^2 g(k, \mathbf{r}, \mathbf{r}') + k^2 g(k, \mathbf{r}, \mathbf{r}')) \mathbf{J}(\mathbf{r}') dv' \\ &= -\mu_0 \mu \iiint_V \delta(\mathbf{r} - \mathbf{r}') \mathbf{J}(\mathbf{r}') dv' = -\mu_0 \mu \mathbf{J}(\mathbf{r}) \end{aligned}$$

by the use of the definition of the delta distribution $\delta(\mathbf{r} - \mathbf{r}')$. Notice that these computations with the delta distribution are formal, but can be made mathematically rigorous. Here, we treat the delta distribution as an "ordinary" function.

It remains to solve the canonical problem (1.43). This problem has spherical symmetry (the source is spherically symmetric), which implies that we are seeking a solution $g(k, \mathbf{r}, \mathbf{r}')$ that only depends on the distance $R = |\mathbf{r} - \mathbf{r}'|$. We translate the source to the origin and we get the problem

$$\nabla^2 g(k, r) + k^2 g(k, r) = -\delta(\mathbf{r}) \quad (1.45)$$

to solve, where $r = |\mathbf{r}|$. For $r \neq 0$, we have

$$\nabla^2 g(k, r) + k^2 g(k, r) = 0$$

The Laplace's operator ∇^2 in spherical coordinates, see Appendix E, implies

$$\frac{1}{r} \frac{d^2}{dr^2} (rg(k, r)) + k^2 g(k, r) = 0$$

or

$$\frac{d^2}{dr^2} (rg(k, r)) + k^2 rg(k, r) = 0$$

with a general solution

$$rg(k, r) = Ae^{ikr} + Be^{-ikr}$$

i.e.,

$$g(k, r) = A \frac{e^{ikr}}{r} + B \frac{e^{-ikr}}{r}$$

Green's function $g(k, r)$, which represents the solution of a unit source at the origin, should, for physical reasons, represent an out-going spherical wave

$$\frac{e^{ikr - i\omega t}}{r}$$

This implies that we choose the constant $B = 0$. The remaining constant A is determined if we integrate (1.45) over a ball of radius ε centered at the origin.²¹

$$\iiint_{r \leq \varepsilon} \nabla^2 g(k, r) \, dv + k^2 \iiint_{r \leq \varepsilon} g(k, r) \, dv = - \iiint_{r \leq \varepsilon} \delta(\mathbf{r}) \, dv = -1$$

Now is

$$\iiint_{r \leq \varepsilon} g(k, r) \, dv = 4\pi A \int_0^\varepsilon \frac{e^{ikr}}{r} r^2 \, dr = 4\pi A \int_0^\varepsilon e^{ikr} r \, dr \rightarrow 0$$

when $\varepsilon \rightarrow 0$. Moreover, the divergence theorem implies

$$\begin{aligned} \iiint_{r \leq \varepsilon} \nabla^2 g(k, r) \, dv &= \iiint_{r \leq \varepsilon} \nabla \cdot \nabla g(k, r) \, dv = \iint_{r=\varepsilon} \hat{\nu} \cdot \nabla g(k, r) \, dS \\ &= \iint_{r=\varepsilon} \frac{dg(k, r)}{dr} \, dS = 4\pi \varepsilon^2 \left. \frac{dg(k, r)}{dr} \right|_{r=\varepsilon} \\ &= 4\pi \varepsilon^2 A e^{ik\varepsilon} \left\{ \frac{ik}{\varepsilon} - \frac{1}{\varepsilon^2} \right\} \rightarrow -4\pi A \end{aligned}$$

when $\varepsilon \rightarrow 0$. Thus, we conclude $-4\pi A = -1$, and $g(k, r) = e^{ikr}/4\pi r$. A translation of the source point back to \mathbf{r}' implies

$$g(k, |\mathbf{r} - \mathbf{r}'|) = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \quad (1.46)$$

²¹A more rigorous way of determining the constant A based upon the theory of distributions is

$$\iiint g(k, r) \nabla^2 \phi(\mathbf{r}) \, dv + k^2 \iiint g(k, r) \phi(\mathbf{r}) \, dv = - \iiint \delta(\mathbf{r}) \phi(\mathbf{r}) \, dv = -\phi(\mathbf{0})$$

where ϕ is an infinitely differentiable function with compact support, *i.e.*, it vanishes outside a finite domain. Use Green's formula on the domain $r \geq \varepsilon$. We get

$$\begin{aligned} -\phi(\mathbf{0}) &= \lim_{\varepsilon \rightarrow 0} \left\{ \iiint_{r \geq \varepsilon} g(k, r) \nabla^2 \phi(\mathbf{r}) \, dv + k^2 \iiint_{r \geq \varepsilon} g(k, r) \phi(\mathbf{r}) \, dv \right\} \\ &= \lim_{\varepsilon \rightarrow 0} \left\{ \iiint_{r \geq \varepsilon} \nabla^2 g(k, r) \phi(\mathbf{r}) \, dv + k^2 \iiint_{r \geq \varepsilon} g(k, r) \phi(\mathbf{r}) \, dv \right\} \\ &\quad + \lim_{\varepsilon \rightarrow 0} \iint_{r=\varepsilon} \left(\frac{\partial g(k, r)}{\partial r} \phi(\mathbf{r}) - \frac{\partial \phi(\mathbf{r})}{\partial r} g(k, r) \right) \, dS \\ &= \lim_{\varepsilon \rightarrow 0} \iint_{r=\varepsilon} \frac{\partial g(k, r)}{\partial r} \phi(\mathbf{r}) \, dS = \lim_{\varepsilon \rightarrow 0} A 4\pi \varepsilon^2 \phi(\mathbf{0}) \frac{e^{ik\varepsilon}}{\varepsilon} \left(ik - \frac{1}{\varepsilon} \right) = -4\pi A \phi(\mathbf{0}) \end{aligned}$$

The solution to the vector potential \mathbf{A} by (1.44) then is

$$\boxed{\mathbf{A}(\mathbf{r}) = \mu_0 \mu \iiint_V \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \mathbf{J}(\mathbf{r}') \, dv'} \quad (1.47)$$

The electric field \mathbf{E} by the use of (1.39) and (1.40) is

$$\boxed{\begin{aligned} \mathbf{E}(\mathbf{r}) &= i\omega \left[\mathbf{A}(\mathbf{r}) + \frac{1}{k^2} \nabla (\nabla \cdot \mathbf{A}(\mathbf{r})) \right] \\ &= ik\eta_0 \eta \left[\mathbf{I}_3 + \frac{1}{k^2} \nabla \nabla \right] \cdot \iiint_V \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \mathbf{J}(\mathbf{r}') \, dv' \end{aligned}} \quad (1.48)$$

where \mathbf{I}_3 is the unit dyadic in three dimensions. The magnetic field \mathbf{H} is by the use of (1.38)

$$\boxed{\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0 \mu} \nabla \times \mathbf{A}(\mathbf{r}) = \nabla \times \iiint_V \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \mathbf{J}(\mathbf{r}') \, dv'} \quad (1.49)$$

Notice that these expressions, hold for an observation point \mathbf{r} both inside and outside the domain V , *i.e.*, outside the region where the currents are non-zero. The interpretation of these integrals for an observation point \mathbf{r} inside the domain V is addressed in Section 1.4.

1.4 The Green's dyadics in isotropic media

The solution of the electromagnetic problem obtained above can also be written in a compact dyadic notation utilizing Green's dyadics. For easy reference, we summarize our canonical problem from above, and define Green's function in free space, $g(k, |\mathbf{r} - \mathbf{r}'|)$, as the solution to

$$\begin{cases} \nabla^2 g(k, |\mathbf{r} - \mathbf{r}'|) + k^2 g(k, |\mathbf{r} - \mathbf{r}'|) = -\delta(\mathbf{r} - \mathbf{r}') \\ \hat{\mathbf{r}} \cdot \nabla g(k, |\mathbf{r} - \mathbf{r}'|) - ik g(k, |\mathbf{r} - \mathbf{r}'|) = o((kr)^{-1}) \end{cases} \quad (1.50)$$

The second condition is the appropriate radiation condition to secure that the radiating solution is obtained. The unique solution to this problem is

$$\boxed{g(k, |\mathbf{r} - \mathbf{r}'|) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}}$$

1.4.1 Full Green's dyadics in free space

We start by formulating the canonical problem (1.50) in a dyadic notion. The definition of the full Green's dyadic is

$$\boxed{\mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) = \mathbf{I}_3 g(k, |\mathbf{r} - \mathbf{r}'|) = \mathbf{I}_3 \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}} \quad (1.51)$$

and it is the unique solution of

$$\begin{cases} \nabla^2 \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) + k^2 \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) = -\mathbf{I}_3 \delta(\mathbf{r} - \mathbf{r}') \\ \nabla \times \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) - ik\hat{\mathbf{r}} \times \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) = o((kr)^{-1}) \end{cases} \quad (1.52)$$

The second condition is the appropriate radiation condition to secure that the radiating solution is obtained.²²

The dyadic $\mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|)$ is symmetric as seen from its definition. This means that

$$\hat{\mathbf{e}}_i \cdot \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) \cdot \hat{\mathbf{e}}_j = \hat{\mathbf{e}}_j \cdot \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) \cdot \hat{\mathbf{e}}_i$$

for any orthonormal basis $\hat{\mathbf{e}}_i$, $i = 1, 2, 3$. However, the curl of Green's dyadic is anti-symmetric as seen from the following calculations:

$$\begin{aligned} \hat{\mathbf{e}}_i \cdot (\nabla \times \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|)) \cdot \hat{\mathbf{e}}_j &= \hat{\mathbf{e}}_i \cdot (\nabla g(k, |\mathbf{r} - \mathbf{r}'|) \times \hat{\mathbf{e}}_j) \\ &= -\hat{\mathbf{e}}_j \cdot (\nabla g(k, |\mathbf{r} - \mathbf{r}'|) \times \hat{\mathbf{e}}_i) \\ &= -\hat{\mathbf{e}}_j \cdot (\nabla \times \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|)) \cdot \hat{\mathbf{e}}_i \end{aligned}$$

This property is written as

$$(\nabla \times \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|))^t = -\nabla \times \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|)$$

Furthermore, due to the spatial dependence $|\mathbf{r} - \mathbf{r}'|$, we have

$$\nabla \times \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) = -\nabla' \times \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|)$$

1.4.2 Green's dyadic for the electric field in free space

The dyadic $\mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|)$ in (1.51) is not divergence-free. To make it useful in solving electromagnetic problems in source-free regions, we need to subtract the part of $\mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|)$ that contains the non-zero divergence contributions.

It is also interesting to investigate Green's dyadic for the electric field, which we denote \mathbf{G}_e . This dyadic is defined as the unique solution of

$$\begin{cases} \nabla \times (\nabla \times \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}')) - k^2 \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}') = \mathbf{I}_3 \delta(\mathbf{r} - \mathbf{r}') \\ \nabla \times \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}') - ik\hat{\mathbf{r}} \times \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}') = o((kr)^{-1}) \end{cases} \quad (1.53)$$

²²The right-hand side is a dyadic-valued quantity and the o -symbol means that each Cartesian component behaves as $o((kr)^{-1})$.

Notice the similarity with the equation for the electric field, see (1.37). The full Green's dyadic $\mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) = \mathbf{I}_3 g(k, |\mathbf{r} - \mathbf{r}'|)$ satisfies (1.52).

$$\nabla^2 \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) + k^2 \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) = -\mathbf{I}_3 \delta(\mathbf{r} - \mathbf{r}')$$

Adding these two equations and using (1.41) give

$$\nabla \times (\nabla \times (\mathbf{G}_e - \mathbf{G})) - k^2 (\mathbf{G}_e - \mathbf{G}) = -\nabla (\nabla \cdot \mathbf{G}) = -\nabla \nabla g$$

which we write as

$$\nabla \times \left(\nabla \times \left(\mathbf{G}_e - \mathbf{G} - \frac{1}{k^2} \nabla \nabla g \right) \right) - k^2 \left(\mathbf{G}_e - \mathbf{G} - \frac{1}{k^2} \nabla \nabla g \right) = \mathbf{0}$$

One solution of this equation is²³

$$\boxed{\mathbf{G}_e = \mathbf{G} + \frac{1}{k^2} \nabla \nabla g = \left(\mathbf{I}_3 + \frac{1}{k^2} \nabla \nabla \right) g} \quad (1.54)$$

From this representation we see that not only \mathbf{G} is a symmetric dyadic but also \mathbf{G}_e . Furthermore, its spatial dependence is $\mathbf{r} - \mathbf{r}'$. Moreover,

$$\begin{aligned} \nabla \times \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}') &= \nabla \times \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) = -\nabla' \times \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) \\ &= -\nabla' \times \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}') \end{aligned}$$

Similar arguments, as for the full Green's dyadic, show that $\nabla \times \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}')$ is an anti-symmetric dyadic. We collect the results.

$$\left\{ \begin{aligned} \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}') &= \left(\mathbf{I}_3 + \frac{1}{k^2} \nabla \nabla \right) g(k, |\mathbf{r} - \mathbf{r}'|) \\ &= \frac{1}{k^2} \{ \nabla \times (\nabla \times \mathbf{I}_3 g(k, |\mathbf{r} - \mathbf{r}'|)) - \mathbf{I}_3 \delta(\mathbf{r} - \mathbf{r}') \} \\ (\mathbf{G}_e(k, \mathbf{r} - \mathbf{r}'))^t &= \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}') \\ (\nabla \times \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}'))^t &= -\nabla \times \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}') \end{aligned} \right.$$

The alternative form of $\mathbf{G}_e(k, \mathbf{r} - \mathbf{r}')$ utilizes the identity

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{I}_3 g(k, |\mathbf{r} - \mathbf{r}'|)) &= \nabla \nabla g(k, |\mathbf{r} - \mathbf{r}'|) - \mathbf{I}_3 \nabla^2 g(k, |\mathbf{r} - \mathbf{r}'|) \\ &= \nabla \nabla g(k, |\mathbf{r} - \mathbf{r}'|) + \mathbf{I}_3 (k^2 g(k, |\mathbf{r} - \mathbf{r}'|) + \delta(\mathbf{r} - \mathbf{r}')) \end{aligned}$$

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²³This is the only solution provided the radiation conditions are satisfied.

- *1.1 The current density $\mathbf{J}(\mathbf{r}, \omega)$ is located inside a ball of radius a . The current density has only a radial component and the components depend only on the radial distance r , *i.e.*,

$$\mathbf{J}(\mathbf{r}, \omega) = \begin{cases} \hat{\mathbf{r}}f(r, \omega) & r \leq a \\ \mathbf{0} & r > a \end{cases}$$

Determine the electric and magnetic fields outside the source region, *i.e.*, for $r > a$. The sources are assumed to be located in vacuum.

Note: A current density, as the one described in the problem, can be generated in a sudden separation of charges, *e.g.*, in a nuclear explosion.

- *1.2 The current density for a electric dipole with the strength p oriented along the $\hat{\mathbf{z}}$ -axis is

$$\mathbf{J}(\mathbf{r}) = -i\omega p \hat{\mathbf{z}} \delta(\mathbf{r})$$

Determine the electric and the magnetic fields outside the dipole, *i.e.*, in the region $r > 0$. Moreover, determine the power density (Poynting's vector) in the domain outside the dipole and the total power P that is propagating through a spherical surface centered at the dipole. The domain outside the dipole is assumed to be vacuum ($\epsilon = \mu = 1$).

Spherical vector waves

There is a special interest in finding solutions of the Maxwell equations in source-free, homogeneous, isotropic materials in the spherical coordinate system (r, θ, ϕ) . The reason for this is at least twofold; 1) we aim at developing efficient tools to solve scattering problems with spherical symmetries, and 2) outside the circumscribing sphere of the scatterer, the fields are naturally expanded in vector waves with spherical symmetries. For this reason, we study solutions to the source-free Maxwell equations

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{r}, \omega) = ik\eta_0\eta\mathbf{H}(\mathbf{r}, \omega) \\ \eta_0\eta\nabla \times \mathbf{H}(\mathbf{r}, \omega) = -ik\mathbf{E}(\mathbf{r}, \omega) \end{cases}$$

in the spherical coordinate system (r, θ, ϕ) . The material parameters, explicitly given by the wave number k and the wave impedance η , are assumed to be constants (may depend on the angular frequency ω). In the general case, these constants can be complex numbers, but in many applications they are real-valued. This latter situation occurs when finding solutions outside the circumscribing sphere of the scatterer in a lossless surrounding material.

We eliminate the field \mathbf{H} , and we get

$$\nabla \times (\nabla \times \mathbf{E}(\mathbf{r})) - k^2\mathbf{E}(\mathbf{r}) = \mathbf{0} \quad (2.1)$$

where, as above, $k^2 = k_0\epsilon\mu$. This equation is the starting point for the development of the special solutions that we are aiming at in this chapter.

In Section 2.1, we motivate the particular form of the vector-valued solutions to the vector Helmholtz' equation used in this textbook. These solutions — spherical vector waves — are formally defined in Section 2.2. Some of the properties of the solutions are analyzed in Section 2.3. The remaining sections in this the chapter contains some results and consequences that are used in the subsequent chapters. Specifically, Section 2.5 and The expansion of the electric Green's dyadic in spherical vector waves is finally derived in Section 2.7.

2.1 Preparatory discussions

Our goal in this section is to construct a set of vector-valued solution to the Maxwell equations that can serve as a basis for the construction of more general solutions to electromagnetic scattering problems.

To find a solution in a spherical region, we utilize the vector spherical harmonics. They are, see Appendix D.3

$$\mathbf{A}_{\tau lm}(\hat{\mathbf{r}}) = \mathbf{A}_n(\hat{\mathbf{r}}) \quad \begin{cases} \tau = 1, 2, 3 \\ l = 0, 1, 2, 3, \dots \\ m = -l, -l+1, \dots, -1, 0, 1, \dots, l-1, l \end{cases}$$

The index n is a multi-index that consists of three different indices, *i.e.*, $n = \tau lm$, where $\tau = 1, 2, 3$, $l = 0, 1, 2, \dots$, and $m = -l, -l+1, \dots, -1, 0, 1, \dots, l-1, l$. The functions are defined and described in Appendix D.3. Their definitions are, see (D.5)

$$\begin{cases} \mathbf{A}_{1lm}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}} \nabla \times (\mathbf{r} Y_{lm}(\hat{\mathbf{r}})) = \frac{1}{\sqrt{l(l+1)}} \nabla Y_{lm}(\hat{\mathbf{r}}) \times \mathbf{r} \\ \mathbf{A}_{2lm}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}} r \nabla Y_{lm}(\hat{\mathbf{r}}) \\ \mathbf{A}_{3lm}(\hat{\mathbf{r}}) = \hat{\mathbf{r}} Y_{lm}(\hat{\mathbf{r}}) \end{cases}$$

and with a change in the exponential of the spherical harmonics, see (D.6)

$$\begin{cases} \mathbf{A}_{1lm}^\dagger(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}} \nabla \times (\mathbf{r} Y_{lm}^\dagger(\hat{\mathbf{r}})) = \frac{1}{\sqrt{l(l+1)}} \nabla Y_{lm}^\dagger(\hat{\mathbf{r}}) \times \mathbf{r} \\ \mathbf{A}_{2lm}^\dagger(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}} r \nabla Y_{lm}^\dagger(\hat{\mathbf{r}}) \\ \mathbf{A}_{3lm}^\dagger(\hat{\mathbf{r}}) = \hat{\mathbf{r}} Y_{lm}^\dagger(\hat{\mathbf{r}}) \end{cases}$$

For $l = 0$ the vector spherical harmonics, $\mathbf{A}_{100}(\hat{\mathbf{r}}) = \mathbf{A}_{200}(\hat{\mathbf{r}}) = \mathbf{0}$, by definition, and only $\mathbf{A}_{300}(\hat{\mathbf{r}}) \neq \mathbf{0}$. The vector spherical harmonics are orthonormal on the unit sphere, *i.e.*,

$$\iint_{\Omega} \mathbf{A}_n(\hat{\mathbf{r}}) \cdot \mathbf{A}_{n'}^\dagger(\hat{\mathbf{r}}) d\Omega = \iint_{\Omega} \mathbf{A}_{\tau lm}(\hat{\mathbf{r}}) \cdot \mathbf{A}_{\tau' l' m'}^\dagger(\hat{\mathbf{r}}) d\Omega = \delta_{\tau, \tau'} \delta_{l, l'} \delta_{m, m'} = \delta_{n, n'}$$

where the surface measure on the unit sphere Ω is $d\Omega = \sin \theta d\theta d\phi$. Other important properties of the vector spherical harmonics are:

$$\begin{cases} \hat{\mathbf{r}} \cdot \mathbf{A}_{\tau lm}(\hat{\mathbf{r}}) = 0, & \tau = 1, 2 \\ \hat{\mathbf{r}} \times \mathbf{A}_{3lm}(\hat{\mathbf{r}}) = \mathbf{0} \end{cases} \quad \begin{cases} \mathbf{A}_{1lm}(\hat{\mathbf{r}}) = \mathbf{A}_{2lm}(\hat{\mathbf{r}}) \times \hat{\mathbf{r}} \\ \mathbf{A}_{2lm}(\hat{\mathbf{r}}) = \hat{\mathbf{r}} \times \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \end{cases} \quad (2.2)$$

More details about the vector spherical harmonics, $\mathbf{A}_{\tau lm}(\hat{\mathbf{r}})$, are collected in Appendix D.3. Moreover, every square integrable vector-valued function, $\mathbf{F}(\hat{\mathbf{r}})$,

defined on the unit sphere has a convergent generalized Fourier expansion in the vector spherical harmonics

$$\mathbf{F}(\hat{\mathbf{r}}) = \sum_{\tau=1}^3 \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{\tau lm} \mathbf{A}_{\tau lm}(\hat{\mathbf{r}}), \quad \theta \in [0, \pi], \phi \in [0, 2\pi)$$

where the (Fourier) coefficients $a_{\tau lm}$ are determined by the integrals

$$a_{\tau lm} = \iint_{\Omega} \mathbf{F}(\hat{\mathbf{r}}) \cdot \mathbf{A}_{\tau lm}^{\dagger}(\hat{\mathbf{r}}) \, d\Omega$$

We return to the original problem of this section, and suppose the Maxwell equations are satisfied in a region in space bounded by two concentric spheres of radii $r_1 < r_2$, *i.e.*, in the region $r_1 \leq r \leq r_2$, where the radii $0 \leq r_1 < r_2 \leq \infty$. If $r_1 = 0$, the region is a sphere of radius r_2 , and if $r_2 = \infty$, then the region is the volume outside the sphere of radius r_1 . To find a solution in this region, we utilize the completeness of the vector spherical harmonics $\mathbf{A}_{\tau lm}(\hat{\mathbf{r}})$, and expand the electric field for a fixed radius $r_1 \leq r \leq r_2$ as

$$\mathbf{E}(\mathbf{r}) = \sum_{lm} \{f_{1lm}(r) \mathbf{A}_{1lm}(\hat{\mathbf{r}}) + f_{2lm}(r) \mathbf{A}_{2lm}(\hat{\mathbf{r}}) + f_{3lm}(r) \mathbf{A}_{3lm}(\hat{\mathbf{r}})\}$$

The expansion coefficients $f_{\tau lm}$ depend in general on r , since we expect different expansion coefficients for different r .

Assuming the series converges uniformly in the angular variables, we can differentiate inside the summation sign. We are helped by (D.14) on page 93

$$\begin{cases} \nabla \times \mathbf{A}_{1lm}(\hat{\mathbf{r}}) = \frac{\mathbf{A}_{2lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \mathbf{A}_{3lm}(\hat{\mathbf{r}})}{r} \\ \nabla \times \mathbf{A}_{2lm}(\hat{\mathbf{r}}) = -\frac{\mathbf{A}_{1lm}(\hat{\mathbf{r}})}{r} \\ \nabla \times \mathbf{A}_{3lm}(\hat{\mathbf{r}}) = \frac{\sqrt{l(l+1)} \mathbf{A}_{1lm}(\hat{\mathbf{r}})}{r} \end{cases}$$

and (2.2), which imply

$$\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r}) &= \sum_{lm} \left(\nabla \times (f_{1lm}(r) \mathbf{A}_{1lm}(\hat{\mathbf{r}}) + f_{2lm}(r) \mathbf{A}_{2lm}(\hat{\mathbf{r}}) + f_{3lm}(r) \mathbf{A}_{3lm}(\hat{\mathbf{r}})) \right) \\
&= \sum_{lm} \left(f'_{1lm}(r) \mathbf{A}_{2lm}(\hat{\mathbf{r}}) - f'_{2lm}(r) \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \right. \\
&\quad \left. + f_{1lm}(r) \frac{\mathbf{A}_{2ml}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \mathbf{A}_{3lm}(\hat{\mathbf{r}})}{r} \right. \\
&\quad \left. - f_{2lm}(r) \frac{\mathbf{A}_{1ml}(\hat{\mathbf{r}})}{r} + f_{3lm}(r) \frac{\sqrt{l(l+1)} \mathbf{A}_{1ml}(\hat{\mathbf{r}})}{r} \right) \\
&= \sum_{lm} \left(\frac{(rf_{1lm}(r))'}{r} \mathbf{A}_{2lm}(\hat{\mathbf{r}}) - \frac{(rf_{2lm}(r))' - f_{3lm}(r) \sqrt{l(l+1)}}{r} \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \right. \\
&\quad \left. + \frac{f_{1lm}(r) \sqrt{l(l+1)}}{r} \mathbf{A}_{3ml}(\hat{\mathbf{r}}) \right)
\end{aligned}$$

Another application of the curl operator and insertion in (2.1) imply

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{E}(\mathbf{r})) - k^2 \mathbf{E}(\mathbf{r}) &= \sum_{lm} \left(- \left(\frac{(rf_{1lm}(r))'}{r} \right)' \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \right. \\
&\quad - \left(\frac{(rf_{2lm}(r))' - f_{3lm}(r) \sqrt{l(l+1)}}{r} \right)' \mathbf{A}_{2lm}(\hat{\mathbf{r}}) - \frac{(rf_{1lm}(r))'}{r^2} \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \\
&\quad - \frac{(rf_{2lm}(r))' - f_{3lm}(r) \sqrt{l(l+1)}}{r^2} \left(\mathbf{A}_{2lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \mathbf{A}_{3lm}(\hat{\mathbf{r}}) \right) \\
&\quad \left. + l(l+1) \frac{f_{1lm}(r)}{r^2} \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \right) \\
&= k^2 \sum_{lm} \{ f_{1lm}(r) \mathbf{A}_{1lm}(\hat{\mathbf{r}}) + f_{2lm}(r) \mathbf{A}_{2lm}(\hat{\mathbf{r}}) + f_{3lm}(r) \mathbf{A}_{3lm}(\hat{\mathbf{r}}) \} = \mathbf{0}
\end{aligned}$$

Since the vector spherical harmonics are orthogonal, each set of index has to satisfy (suppress the indices n)

$$\begin{cases} \frac{d}{dr} \left(r^2 \frac{d}{dr} f_1(r) \right) - l(l+1) f_1(r) + k^2 r^2 f_1(r) = 0 \\ \frac{d}{dr} \left(r^2 \frac{d}{dr} f_2(r) \right) - r \sqrt{l(l+1)} \frac{d}{dr} f_3(r) + k^2 r^2 f_2(r) = 0 \\ \sqrt{l(l+1)} \frac{d}{dr} (r f_2(r)) - l(l+1) f_3(r) + k^2 r^2 f_3(r) = 0 \end{cases} \quad (2.3)$$

We observe that the first function f_{1lm} does not couple to the other two functions f_{2lm} and f_{3lm} . However, f_{2lm} and f_{3lm} couple to each other, and they depend on

each other. In fact, the following relation between $f_2(r)$ and $f_3(r)$ in (2.3) eliminates $f_2(r)$ in the second and third equations:

$$f_2(r) = \frac{\frac{d}{dr}(r^2 f_3(r))}{r\sqrt{l(l+1)}}$$

since then the equations become

$$\begin{cases} \frac{d}{dr} \left(r^2 \frac{d}{dr} \frac{\frac{d}{dr}(r^2 f_3(r))}{r} \right) - rl(l+1) \frac{df_3(r)}{dr} + k^2 r^2 \frac{\frac{d}{dr}(r^2 f_3(r))}{r} = 0 \\ r^2 f_3''(r) + 4r f_3'(r) + 2f_3(r) - l(l+1)f_3(r) + k^2 r^2 f_3(r) = 0 \end{cases}$$

or

$$\begin{cases} \frac{d}{dr} (r^2 f_3''(r) + 4r f_3'(r) + 2f_3(r) - l(l+1)f_3(r) + k^2 r^2 f_3(r)) = 0 \\ r^2 f_3''(r) + 4r f_3'(r) + 2f_3(r) - l(l+1)f_3(r) + k^2 r^2 f_3(r) = 0 \end{cases}$$

The first equation in (2.3) is the differential equation for spherical Bessel functions, *i.e.*,

$$z^2 \frac{d^2}{dz^2} Z_l(z) + 2z \frac{d}{dz} Z_l(z) + (z^2 - l(l+1)) Z_l(z) = 0$$

Details about these functions are listed in Appendix B.2. It is also convenient to identify the differential equation for a spherical Bessel function times a power, *i.e.*, $F_{nl}(z) = z^n Z_l(z)$, where $n = 0, \pm 1, \pm 2, \dots$ is an integer. The function $F_{nl}(z)$ satisfies, see (B.24)

$$z^2 \frac{d^2}{dz^2} F_{nl}(z) + 2z(1-n) \frac{d}{dz} F_{nl}(z) + (z^2 - l(l+1) + n(n+1) - 2n) F_{nl}(z) = 0$$

From this equation we see that $f_3(r) = F_{-1l}(kr)$.

We summarize, the radial functions have to be a combination of

$$\begin{cases} f_{1lm}(r) = j_l(kr) \text{ and/or } h_l^{(1)}(kr) \\ f_{2lm}(r) = \frac{(kr j_l(kr))'}{kr} \text{ and/or } \frac{(kr h_l^{(1)}(kr))'}{kr} \\ f_{3lm}(r) = \sqrt{l(l+1)} \frac{j_l(kr)}{kr} \text{ and/or } \sqrt{l(l+1)} \frac{h_l^{(1)}(kr)}{kr} \end{cases}$$

where the solutions f_{2lm} and f_{3lm} are tied together. Notice that the prime here denotes differentiation w.r.t. the whole argument of the functions, *i.e.*, kr . Another combination used in the scattering formulation of antennas is

$$\begin{cases} f_{1lm}(r) = h_l^{(1)}(kr) \text{ and/or } h_l^{(2)}(kr) \\ f_{2lm}(r) = \frac{(kr h_l^{(1)}(kr))'}{kr} \text{ and/or } \frac{(kr h_l^{(2)}(kr))'}{kr} \\ f_{3lm}(r) = \sqrt{l(l+1)} \frac{h_l^{(1)}(kr)}{kr} \text{ and/or } \sqrt{l(l+1)} \frac{h_l^{(2)}(kr)}{kr} \end{cases}$$

We are now ready to define two sets of spherical vector solutions to the Maxwell equations in spherical regions.

2.2 Definition of spherical vector waves

Following the results in Section 2.1, we define the out-going, or radiating¹, spherical vector waves, $\mathbf{u}_{\tau lm}(k\mathbf{r})$ as

$$\begin{cases} \mathbf{u}_{1lm}(k\mathbf{r}) = h_l^{(1)}(kr) \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \\ \mathbf{u}_{2lm}(k\mathbf{r}) = \frac{(krh_l^{(1)}(kr))'}{kr} \mathbf{A}_{2lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \frac{h_l^{(1)}(kr)}{kr} \mathbf{A}_{3lm}(\hat{\mathbf{r}}) \end{cases} \quad (2.4)$$

the in-going spherical vector waves, $\mathbf{w}_{\tau lm}(k\mathbf{r})$ as

$$\begin{cases} \mathbf{w}_{1lm}(k\mathbf{r}) = h_l^{(2)}(kr) \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \\ \mathbf{w}_{2lm}(k\mathbf{r}) = \frac{(krh_l^{(2)}(kr))'}{kr} \mathbf{A}_{2lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \frac{h_l^{(2)}(kr)}{kr} \mathbf{A}_{3lm}(\hat{\mathbf{r}}) \end{cases} \quad (2.5)$$

and regular spherical vector waves $\mathbf{v}_{\tau lm}(k\mathbf{r})$ as

$$\begin{cases} \mathbf{v}_{1lm}(k\mathbf{r}) = j_l(kr) \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \\ \mathbf{v}_{2lm}(k\mathbf{r}) = \frac{(krj_l(kr))'}{kr} \mathbf{A}_{2lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \frac{j_l(kr)}{kr} \mathbf{A}_{3lm}(\hat{\mathbf{r}}) \end{cases} \quad (2.6)$$

The regular and in- and out-going spherical vector waves are very similar — they only differ by the spherical Bessel or Hankel functions. Note that for $l = 0$, $\mathbf{u}_{100}(k\mathbf{r}) = \mathbf{u}_{200}(k\mathbf{r}) = \mathbf{w}_{100}(k\mathbf{r}) = \mathbf{w}_{200}(k\mathbf{r}) = \mathbf{v}_{100}(k\mathbf{r}) = \mathbf{v}_{200}(k\mathbf{r}) = \mathbf{0}$, so in all cases the indices l and m assume the following values:

$$l = 1, 2, 3, \dots, \quad m = -l, -l+1, \dots, -1, 0, 1, \dots, l-1, l$$

By construction, both these sets forms a complete set on a spherical surface, and these sets satisfy

$$\begin{cases} \nabla \times (\nabla \times \mathbf{u}_n(k\mathbf{r})) = k^2 \mathbf{u}_n(k\mathbf{r}) \\ \nabla \times (\nabla \times \mathbf{w}_n(k\mathbf{r})) = k^2 \mathbf{w}_n(k\mathbf{r}) \\ \nabla \times (\nabla \times \mathbf{v}_n(k\mathbf{r})) = k^2 \mathbf{v}_n(k\mathbf{r}) \end{cases} \quad (2.7)$$

and

$$\begin{cases} \nabla \cdot \mathbf{u}_n(k\mathbf{r}) = 0 \\ \nabla \cdot \mathbf{w}_n(k\mathbf{r}) = 0 \\ \nabla \cdot \mathbf{v}_n(k\mathbf{r}) = 0 \end{cases}$$

where we have adopted the multi-index n . Moreover, we find by direct calculations

$$\begin{aligned} \nabla \times \mathbf{v}_{1lm}(k\mathbf{r}) &= k j_l'(kr) \mathbf{A}_{2lm}(\hat{\mathbf{r}}) + j_l(kr) \frac{\mathbf{A}_{2ml}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \mathbf{A}_{3lm}(\hat{\mathbf{r}})}{r} \\ &= k \mathbf{v}_{2lm}(k\mathbf{r}) \end{aligned}$$

¹Below, in Section 2.6, the rationale for the name out-going or radiating spherical vector waves is given.

and

$$\nabla \times \mathbf{v}_{2lm}(k\mathbf{r}) = \frac{1}{k} \nabla \times (\nabla \times \mathbf{v}_{1lm}(k\mathbf{r})) = k\mathbf{v}_{1lm}(k\mathbf{r})$$

This way we have showed that

$$\begin{cases} \mathbf{v}_{1lm}(k\mathbf{r}) = \frac{1}{k} \nabla \times \mathbf{v}_{2lm}(k\mathbf{r}) \\ \mathbf{v}_{2lm}(k\mathbf{r}) = \frac{1}{k} \nabla \times \mathbf{v}_{1lm}(k\mathbf{r}) \end{cases} \quad (2.8)$$

and we have an alternative definition of the regular spherical vector waves, $\mathbf{v}_{\tau lm}(k\mathbf{r})$

$$\begin{cases} \mathbf{v}_{1lm}(k\mathbf{r}) = j_l(kr) \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \\ \mathbf{v}_{2lm}(k\mathbf{r}) = \frac{1}{k} \nabla \times (j_l(kr) \mathbf{A}_{1lm}(\hat{\mathbf{r}})) \end{cases} \quad (2.9)$$

A similar relation holds for the in- and out-going or radiating spherical vector waves, and we get

$$\begin{cases} \mathbf{u}_{1lm}(k\mathbf{r}) = h_l^{(1)}(kr) \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \\ \mathbf{u}_{2lm}(k\mathbf{r}) = \frac{1}{k} \nabla \times (h_l^{(1)}(kr) \mathbf{A}_{1lm}(\hat{\mathbf{r}})) \end{cases} \quad (2.10)$$

and

$$\begin{cases} \mathbf{w}_{1lm}(k\mathbf{r}) = h_l^{(2)}(kr) \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \\ \mathbf{w}_{2lm}(k\mathbf{r}) = \frac{1}{k} \nabla \times (h_l^{(2)}(kr) \mathbf{A}_{1lm}(\hat{\mathbf{r}})) \end{cases} \quad (2.11)$$

The functions in (2.4)–(2.6) have zero divergence as stated above. These functions are often augmented by a third spherical vector wave, $\tau = 3$, that has non-zero divergence, and therefore do not enter in the expansions of the electric field in a isotropic, homogeneous region. This type of wave is called the longitudinal (irrotational) spherical vector waves, as opposed to the ones defined above, $\tau = 1, 2$, in (2.4) and (2.6), which are called the transverse spherical vector waves. To make the list complete, we repeat the definitions above, and get

$$\begin{cases} \mathbf{u}_{1lm}(k\mathbf{r}) = h_l^{(1)}(kr) \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \\ \mathbf{u}_{2lm}(k\mathbf{r}) = \frac{(kr h_l^{(1)}(kr))'}{kr} \mathbf{A}_{2lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \frac{h_l^{(1)}(kr)}{kr} \mathbf{A}_{3lm}(\hat{\mathbf{r}}) \\ \mathbf{u}_{3lm}(k\mathbf{r}) = h_l^{(1)'}(kr) \mathbf{A}_{3lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \frac{h_l^{(1)}(kr)}{kr} \mathbf{A}_{2lm}(\hat{\mathbf{r}}) = \frac{1}{k} \nabla \cdot (h_l^{(1)}(kr) \mathbf{Y}_{lm}(\hat{\mathbf{r}})) \end{cases}$$

$$\begin{cases} \mathbf{w}_{1lm}(k\mathbf{r}) = h_l^{(2)}(kr) \mathbf{A}_{1lm}(\hat{\mathbf{r}}) \\ \mathbf{w}_{2lm}(k\mathbf{r}) = \frac{(kr h_l^{(2)}(kr))'}{kr} \mathbf{A}_{2lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \frac{h_l^{(2)}(kr)}{kr} \mathbf{A}_{3lm}(\hat{\mathbf{r}}) \\ \mathbf{w}_{3lm}(k\mathbf{r}) = h_l^{(2)'}(kr) \mathbf{A}_{3lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \frac{h_l^{(2)}(kr)}{kr} \mathbf{A}_{2lm}(\hat{\mathbf{r}}) = \frac{1}{k} \nabla \cdot (h_l^{(2)}(kr) \mathbf{Y}_{lm}(\hat{\mathbf{r}})) \end{cases}$$

and regular spherical vector waves $\mathbf{v}_{\tau lm}(k\mathbf{r})$ as

$$\begin{cases} \mathbf{v}_{1lm}(k\mathbf{r}) = j_l(kr)\mathbf{A}_{1lm}(\hat{\mathbf{r}}) \\ \mathbf{v}_{2lm}(k\mathbf{r}) = \frac{(krj_l(kr))'}{kr}\mathbf{A}_{2lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)}\frac{j_l(kr)}{kr}\mathbf{A}_{3lm}(\hat{\mathbf{r}}) \\ \mathbf{v}_{3lm}(k\mathbf{r}) = j_l'(kr)\mathbf{A}_{3lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)}\frac{j_l(kr)}{kr}\mathbf{A}_{2lm}(\hat{\mathbf{r}}) = \frac{1}{k}\nabla(j_l(kr)Y_{lm}(\hat{\mathbf{r}})) \end{cases}$$

and the corresponding spherical vector waves with the vector spherical harmonics $\mathbf{A}_n^\dagger(\hat{\mathbf{r}})$, *viz.*

$$\begin{cases} \mathbf{u}_{1lm}^\dagger(k\mathbf{r}) = h_l^{(1)}(kr)\mathbf{A}_{1lm}^\dagger(\hat{\mathbf{r}}) \\ \mathbf{u}_{2lm}^\dagger(k\mathbf{r}) = \frac{(krh_l^{(1)}(kr))'}{kr}\mathbf{A}_{2lm}^\dagger(\hat{\mathbf{r}}) + \sqrt{l(l+1)}\frac{h_l^{(1)}(kr)}{kr}\mathbf{A}_{3lm}^\dagger(\hat{\mathbf{r}}) \\ \mathbf{u}_{3lm}^\dagger(k\mathbf{r}) = h_l^{(1)'}(kr)\mathbf{A}_{3lm}^\dagger(\hat{\mathbf{r}}) + \sqrt{l(l+1)}\frac{h_l^{(1)}(kr)}{kr}\mathbf{A}_{2lm}^\dagger(\hat{\mathbf{r}}) \end{cases}$$

$$\begin{cases} \mathbf{w}_{1lm}^\dagger(k\mathbf{r}) = h_l^{(2)}(kr)\mathbf{A}_{1lm}^\dagger(\hat{\mathbf{r}}) \\ \mathbf{w}_{2lm}^\dagger(k\mathbf{r}) = \frac{(krh_l^{(2)}(kr))'}{kr}\mathbf{A}_{2lm}^\dagger(\hat{\mathbf{r}}) + \sqrt{l(l+1)}\frac{h_l^{(2)}(kr)}{kr}\mathbf{A}_{3lm}^\dagger(\hat{\mathbf{r}}) \\ \mathbf{w}_{3lm}^\dagger(k\mathbf{r}) = h_l^{(2)'}(kr)\mathbf{A}_{3lm}^\dagger(\hat{\mathbf{r}}) + \sqrt{l(l+1)}\frac{h_l^{(2)}(kr)}{kr}\mathbf{A}_{2lm}^\dagger(\hat{\mathbf{r}}) \end{cases}$$

$$\begin{cases} \mathbf{v}_{1lm}^\dagger(k\mathbf{r}) = j_l(kr)\mathbf{A}_{1lm}^\dagger(\hat{\mathbf{r}}) \\ \mathbf{v}_{2lm}^\dagger(k\mathbf{r}) = \frac{(krj_l(kr))'}{kr}\mathbf{A}_{2lm}^\dagger(\hat{\mathbf{r}}) + \sqrt{l(l+1)}\frac{j_l(kr)}{kr}\mathbf{A}_{3lm}^\dagger(\hat{\mathbf{r}}) \\ \mathbf{v}_{3lm}^\dagger(k\mathbf{r}) = j_l'(kr)\mathbf{A}_{3lm}^\dagger(\hat{\mathbf{r}}) + \sqrt{l(l+1)}\frac{j_l(kr)}{kr}\mathbf{A}_{2lm}^\dagger(\hat{\mathbf{r}}) \end{cases}$$

Note the difference between the dagger, \dagger , operation and complex conjugate, $*$. We have for real k values and real angles θ and ϕ

$$\begin{cases} \mathbf{v}_n^\dagger(k\mathbf{r}) = \mathbf{v}_n^*(k\mathbf{r}) \\ \mathbf{u}_n^\dagger(k\mathbf{r}) = \mathbf{w}_n^*(k\mathbf{r}) = 2\mathbf{v}_n^\dagger(k\mathbf{r}) - \mathbf{w}_n^\dagger(k\mathbf{r}) \\ \mathbf{w}_n^\dagger(k\mathbf{r}) = \mathbf{u}_n^*(k\mathbf{r}) = 2\mathbf{v}_n^\dagger(k\mathbf{r}) - \mathbf{u}_n^\dagger(k\mathbf{r}) \end{cases} \quad (2.12)$$

2.2.1 Expansions of the fields

The electric field in a source-free region satisfy

$$\nabla \times (\nabla \times \mathbf{E}(\mathbf{r}, \omega)) = k^2 \mathbf{E}(\mathbf{r}, \omega)$$

The electric field outside a sphere that circumscribes the scatterer (antennas or radiating system), therefore, due to completeness of the spherical vector waves, has

a general expansion

$$\begin{aligned} \mathbf{E}(\mathbf{r}, \omega) &= k\sqrt{\eta_0\eta} \sum_{\tau=1}^2 \sum_{l=1}^{\infty} \sum_{m=-l}^l (a_{\tau lm} \mathbf{v}_{\tau lm}(k\mathbf{r}) + f_{\tau lm} \mathbf{u}_{\tau lm}(k\mathbf{r})) \\ &= k\sqrt{\eta_0\eta} \sum_n (a_n \mathbf{v}_n(k\mathbf{r}) + f_n \mathbf{u}_n(k\mathbf{r})) \end{aligned} \quad (2.13)$$

Notice that the summation only includes $\tau = 1, 2$, since the electric field is divergenceless in free space. In general we need an expansion in both regular and out-going spherical vector waves. An extra normalization factor, $k\sqrt{\eta_0\eta}$, is introduced to get simple power expressions (power normalization) below. Note that the notion \sum_n for this multiple sum extends over $\tau = 1, 2$ and it starts at $l = 1$. The expansion of the corresponding magnetic field then is by the use of Faraday's law

$$\begin{aligned} \mathbf{H}(\mathbf{r}, \omega) &= \frac{k\sqrt{\eta_0\eta}}{i\eta_0\eta} \sum_{\tau=1}^2 \sum_{l=1}^{\infty} \sum_{m=-l}^l (a_{\tau lm} \mathbf{v}_{\bar{\tau} lm}(k\mathbf{r}) + f_{\tau lm} \mathbf{u}_{\bar{\tau} lm}(k\mathbf{r})) \\ &= \frac{k}{i\sqrt{\eta_0\eta}} \sum_n (a_n \mathbf{v}_{\bar{n}}(k\mathbf{r}) + f_n \mathbf{u}_{\bar{n}}(k\mathbf{r})) \end{aligned} \quad (2.14)$$

where we have introduced the dual index to τ , defined by $\bar{1} = 2$ and $\bar{2} = 1$, and the short-hand notation $\bar{n} = \{\bar{\tau}lm\}$.

The expansions of the electric and the magnetic fields in (2.13) and (2.14) are the partial wave or multipole expansions of the fields. Every l, m -term in the sum defines a partial wave or multipole of degree (l, m) . From the alternative representations, (2.4) and (2.6), we see that the multipole with $\tau = 1$ has no radial component ($\hat{\mathbf{r}}$ component), since $\hat{\mathbf{r}} \cdot \mathbf{A}_{1lm}(\hat{\mathbf{r}}) = 0$. This τ -index generates a transverse electric (TE) multipole of degree (l, m) . Similarly, $\tau = 2$ generates a transverse magnetic (TM) multipole of degree (l, m) .

An equivalent expansion of the electric and the magnetic fields outside the circumscribing sphere of the scatterer (antennas or radiating system) is obtained by employing the in- and out-going radial combinations instead of the regular and out-going functions.

$$\begin{aligned} \mathbf{E}(\mathbf{r}, \omega) &= k\sqrt{\eta_0\eta} \sum_{\tau=1}^2 \sum_{l=1}^{\infty} \sum_{m=-l}^l (a_{\tau lm} \mathbf{w}_{\tau lm}(k\mathbf{r}) + b_{\tau lm} \mathbf{u}_{\tau lm}(k\mathbf{r})) \\ &= k\sqrt{\eta_0\eta} \sum_n (a_n \mathbf{w}_n(k\mathbf{r}) + b_n \mathbf{u}_n(k\mathbf{r})) \end{aligned}$$

and

$$\begin{aligned} \mathbf{H}(\mathbf{r}, \omega) &= \frac{k\sqrt{\eta_0\eta}}{i\eta_0\eta} \sum_{\tau=1}^2 \sum_{l=1}^{\infty} \sum_{m=-l}^l (a_{\tau lm} \mathbf{w}_{\bar{\tau} lm}(k\mathbf{r}) + b_{\tau lm} \mathbf{u}_{\bar{\tau} lm}(k\mathbf{r})) \\ &= \frac{k}{i\sqrt{\eta_0\eta}} \sum_n (a_n \mathbf{w}_{\bar{n}}(k\mathbf{r}) + b_n \mathbf{u}_{\bar{n}}(k\mathbf{r})) \end{aligned}$$

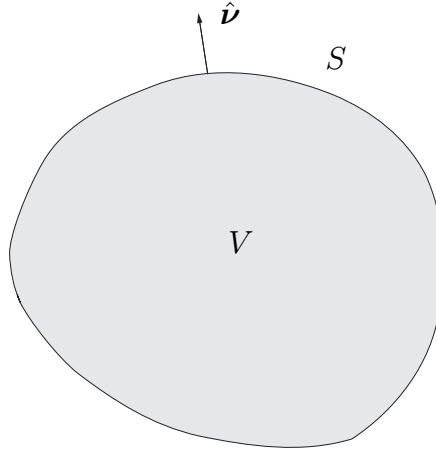


Figure 2.1: The geometry used in the proof of the orthogonality relations in Section 2.3.

2.3 Orthogonality and reciprocity relations

The spherical vector waves were defined in (2.9)–(2.11), or (2.4)–(2.6). Only $\tau = 1, 2$ are used in expansions in source-free regions. For this reason, we first restrict ourselves to this case, and, moreover, the wave number k is assumed to be a real number. In this chapter, we use, as above, a multi-index n for the indices τlm . The dual index \bar{n} is defined as $\bar{n} = \{\bar{\tau} l m\}$, where, as before, $\bar{1} = 2$ and $\bar{2} = 1$.

In this section we derive the orthogonality relations for these waves. For the regular spherical vector waves the relation is:

$$\boxed{\iint_S \left\{ \mathbf{v}_n \times (\nabla \times \mathbf{v}_{n'}^\dagger) - \mathbf{v}_{n'}^\dagger \times (\nabla \times \mathbf{v}_n) \right\} \cdot \hat{\nu} \, dS = 0, \quad \tau, \tau' = 1, 2} \quad (2.15)$$

for all closed, regular surfaces S , see Figure 2.1. To prove this relation we apply the divergence theorem and expand the integrand.

$$\begin{aligned} & \iint_S \left\{ \mathbf{v}_n \times (\nabla \times \mathbf{v}_{n'}^\dagger) - \mathbf{v}_{n'}^\dagger \times (\nabla \times \mathbf{v}_n) \right\} \cdot \hat{\nu} \, dS \\ &= \iiint_V \nabla \cdot \left\{ \mathbf{v}_n \times (\nabla \times \mathbf{v}_{n'}^\dagger) - \mathbf{v}_{n'}^\dagger \times (\nabla \times \mathbf{v}_n) \right\} \, dv \\ &= \iiint_V \left\{ (\nabla \times \mathbf{v}_n) \cdot (\nabla \times \mathbf{v}_{n'}^\dagger) - \mathbf{v}_n \cdot (\nabla \times (\nabla \times \mathbf{v}_{n'}^\dagger)) \right. \\ &\quad \left. - (\nabla \times \mathbf{v}_{n'}^\dagger) \cdot (\nabla \times \mathbf{v}_n) + \mathbf{v}_{n'}^\dagger \cdot (\nabla \times (\nabla \times \mathbf{v}_n)) \right\} \, dv \\ &= \iiint_V \left\{ -k^2 \mathbf{v}_n \cdot \mathbf{v}_{n'}^\dagger + k^2 \mathbf{v}_{n'}^\dagger \cdot \mathbf{v}_n \right\} \, dv = 0 \end{aligned}$$

where V is the volume enclosed by the surface S . Notice that it is essential that the wave number k is a real number in this derivation. An analogous derivation implies the reciprocity relation (replace $\mathbf{v}_{n'}^\dagger$ by $\mathbf{v}_{n'}$)

$$\boxed{\iint_S \{\mathbf{v}_n \times (\nabla \times \mathbf{v}_{n'}) - \mathbf{v}_{n'} \times (\nabla \times \mathbf{v}_n)\} \cdot \hat{\nu} \, dS = 0, \quad \tau, \tau' = 1, 2} \quad (2.16)$$

The identities for combinations of regular and out- and in-going waves are derived in a similar way. If the surface S does not include the origin, $\mathbf{r} = \mathbf{0}$, then, as in the regular wave case ($\tau, \tau' = 1, 2$)

$$\boxed{\begin{aligned} \iint_S \{\mathbf{v}_n \times (\nabla \times \mathbf{u}_{n'}^\dagger) - \mathbf{u}_{n'}^\dagger \times (\nabla \times \mathbf{v}_n)\} \cdot \hat{\nu} \, dS &= 0 \\ \iint_S \{\mathbf{v}_n^\dagger \times (\nabla \times \mathbf{u}_{n'}) - \mathbf{u}_{n'} \times (\nabla \times \mathbf{v}_n^\dagger)\} \cdot \hat{\nu} \, dS &= 0 \end{aligned} \quad \mathbf{0} \notin V} \quad (2.17)$$

and

$$\boxed{\begin{aligned} \iint_S \{\mathbf{v}_n \times (\nabla \times \mathbf{w}_{n'}^\dagger) - \mathbf{w}_{n'}^\dagger \times (\nabla \times \mathbf{v}_n)\} \cdot \hat{\nu} \, dS &= 0 \\ \iint_S \{\mathbf{v}_n^\dagger \times (\nabla \times \mathbf{w}_{n'}) - \mathbf{w}_{n'} \times (\nabla \times \mathbf{v}_n^\dagger)\} \cdot \hat{\nu} \, dS &= 0 \end{aligned} \quad \mathbf{0} \notin V} \quad (2.18)$$

and the reciprocity relation

$$\boxed{\begin{aligned} \iint_S \{\mathbf{v}_n \times (\nabla \times \mathbf{u}_{n'}) - \mathbf{u}_{n'} \times (\nabla \times \mathbf{v}_n)\} \cdot \hat{\nu} \, dS &= 0 \\ \iint_S \{\mathbf{v}_n \times (\nabla \times \mathbf{w}_{n'}) - \mathbf{w}_{n'} \times (\nabla \times \mathbf{v}_n)\} \cdot \hat{\nu} \, dS &= 0 \quad \mathbf{0} \notin V \\ \iint_S \{\mathbf{u}_n \times (\nabla \times \mathbf{w}_{n'}) - \mathbf{w}_{n'} \times (\nabla \times \mathbf{u}_n)\} \cdot \hat{\nu} \, dS &= 0 \end{aligned}}$$

for all closed, regular surfaces S not including the origin.

If, however, the surface S includes the origin, the divergence theorem gives (ϵ is small enough to ensure that the sphere, $r = \epsilon$, lies inside the inscribed sphere of the

scatterer)

$$\begin{aligned}
& \iint_S \left\{ \mathbf{v}_n \times (\nabla \times \mathbf{u}_{n'}^\dagger) - \mathbf{u}_{n'}^\dagger \times (\nabla \times \mathbf{v}_n) \right\} \cdot \hat{\nu} \, dS \\
&= \iint_{r=\epsilon} \left\{ \mathbf{v}_n \times (\nabla \times \mathbf{u}_{n'}^\dagger) - \mathbf{u}_{n'}^\dagger \times (\nabla \times \mathbf{v}_n) \right\} \cdot \hat{\mathbf{r}} \, dS \\
&= k \iint_{r=\epsilon} \left\{ \mathbf{v}_n \times \mathbf{u}_{n'}^\dagger - \mathbf{u}_{n'}^\dagger \times \mathbf{v}_n \right\} \cdot \hat{\mathbf{r}} \, dS = k \iint_{r=\epsilon} \left\{ \mathbf{v}_n \cdot (\mathbf{u}_{n'}^\dagger \times \hat{\mathbf{r}}) - \mathbf{u}_{n'}^\dagger \cdot (\mathbf{v}_n \times \hat{\mathbf{r}}) \right\} \, dS
\end{aligned}$$

The integral over the sphere, $r = \epsilon$, is zero if $\tau \neq \tau'$, since the integrand is proportional to $\mathbf{A}_{\tau lm} \cdot \mathbf{A}_{\tau' l' m'}^\dagger$. Orthogonality then gives the result. For $\tau = \tau' = 1$ we have

$$\begin{aligned}
& \iint_S \left\{ \mathbf{v}_n \times (\nabla \times \mathbf{u}_{n'}^\dagger) - \mathbf{u}_{n'}^\dagger \times (\nabla \times \mathbf{v}_n) \right\} \cdot \hat{\nu} \, dS \\
&= k\epsilon^2 \delta_{ll'} \delta_{mm'} \left\{ j_l(k\epsilon) \frac{(k\epsilon h_l^{(1)}(k\epsilon))'}{k\epsilon} - h_l^{(1)}(k\epsilon) \frac{(k\epsilon j_l(k\epsilon))'}{k\epsilon} \right\} \\
&= k\epsilon^2 \delta_{ll'} \delta_{mm'} \left\{ j_l(k\epsilon) h_l^{(1)'}(k\epsilon) - h_l^{(1)}(k\epsilon) j_l'(k\epsilon) \right\} = \delta_{ll'} \delta_{mm'} \frac{i}{k}
\end{aligned}$$

where we used the Wronskian for the spherical Bessel and Hankel functions

$$j_l(z) h_l^{(1)'}(z) - h_l^{(1)}(z) j_l'(z) = \frac{i}{z^2}$$

Notice that the orthogonality relation is independent of the value of ϵ as expected.

Similarly, for $\tau = \tau' = 2$ we have

$$\iint_S \left\{ \mathbf{v}_n \times (\nabla \times \mathbf{u}_{n'}^\dagger) - \mathbf{u}_{n'}^\dagger \times (\nabla \times \mathbf{v}_n) \right\} \cdot \hat{\nu} \, dS = \delta_{ll'} \delta_{mm'} \frac{i}{k}$$

In summary, $(\tau, \tau' = 1, 2)$

$$\boxed{
\begin{aligned}
& \iint_S \left\{ \mathbf{v}_n \times (\nabla \times \mathbf{u}_{n'}^\dagger) - \mathbf{u}_{n'}^\dagger \times (\nabla \times \mathbf{v}_n) \right\} \cdot \hat{\nu} \, dS = \delta_{nn'} \frac{i}{k} \\
& \iint_S \left\{ \mathbf{v}_n^\dagger \times (\nabla \times \mathbf{u}_{n'}) - \mathbf{u}_{n'} \times (\nabla \times \mathbf{v}_n^\dagger) \right\} \cdot \hat{\nu} \, dS = \delta_{nn'} \frac{i}{k}
\end{aligned}
} \quad \mathbf{0} \in V \quad (2.19)$$

if the surface S encloses the origin.

The out-going and in-going waves have an orthogonality relation similar to the regular one in (2.19).

$$\begin{aligned}
& \iint_S \left\{ \mathbf{u}_n \times (\nabla \times \mathbf{w}_{n'}^\dagger) - \mathbf{w}_{n'}^\dagger \times (\nabla \times \mathbf{u}_n) \right\} \cdot \hat{\nu} \, dS \\
&= k\epsilon^2 \delta_{nn'} \left\{ h_l^{(1)}(k\epsilon) \frac{(k\epsilon h_l^{(2)}(k\epsilon))'}{k\epsilon} - \frac{(k\epsilon h_l^{(1)}(k\epsilon))'}{k\epsilon} h_l^{(2)}(k\epsilon) \right\} = -\delta_{nn'} \frac{2i}{k}
\end{aligned}$$

where we used the Wronskian for the Hankel functions

$$h_l^{(1)}(z)h_l^{(2)'}(z) - h_l^{(2)}(z)h_l^{(1)'}(z) = -\frac{2i}{z^2}$$

We get

$$\boxed{\begin{aligned} \iint_S \left\{ \mathbf{u}_n \times (\nabla \times \mathbf{w}_{n'}^\dagger) - \mathbf{w}_{n'}^\dagger \times (\nabla \times \mathbf{u}_n) \right\} \cdot \hat{\nu} \, dS &= -\delta_{nn'} \frac{2i}{k}, \quad \mathbf{0} \in V \\ \iint_S \left\{ \mathbf{u}_n \times (\nabla \times \mathbf{w}_{n'}^\dagger) - \mathbf{w}_{n'}^\dagger \times (\nabla \times \mathbf{u}_n) \right\} \cdot \hat{\nu} \, dS &= 0, \quad \mathbf{0} \notin V \end{aligned}} \quad (2.20)$$

The reciprocity relation for the out-going waves is

$$\iint_S \left\{ \mathbf{u}_n \times (\nabla \times \mathbf{u}_{n'}) - \mathbf{u}_{n'} \times (\nabla \times \mathbf{u}_n) \right\} \cdot \hat{\nu} \, dS = 0 \quad (2.21)$$

for all regular surfaces S irrespective if S includes the origin or not. This conclusion can be made since the integral is proportional to

$$\frac{(k\epsilon h_l^{(1)}(k\epsilon))'}{k\epsilon} h_l^{(1)}(k\epsilon) - h_l^{(1)}(k\epsilon) \frac{(k\epsilon h_l^{(1)}(k\epsilon))'}{k\epsilon} = 0$$

2.4 Linear independence

A countable, infinite family of functions are linearly independent if every finite subset is linearly independent. More precisely, the system $\{\phi_j\}$ is linearly independent if for every nonempty subset of $\{\phi_j\}$, say $\{\phi_1, \phi_2, \dots, \phi_N\}$, we have

$$\sum_{j=1}^N c_j \phi_j = 0 \quad \Rightarrow \quad c_j = 0$$

Lemma 2.1. *The family $\hat{\nu}(\mathbf{r}) \times \mathbf{u}_{\tau lm}(k\mathbf{r})$ is linearly independent on $\mathbf{L}_t^2(\partial\Omega)$.*

Proof: Consider

$$\sum_{\tau=1}^2 \sum_{l=0}^L \sum_{m=-l}^l c_{\tau lm} \hat{\nu}(\mathbf{r}) \times \mathbf{u}_{\tau lm}(k\mathbf{r}) = \mathbf{0}, \quad \mathbf{r} \in \partial\Omega$$

We have to prove that $c_{\tau lm} = 0$ for all indices.

In the region exterior to Ω define the function

$$\mathbf{u}(\mathbf{r}) = \sum_{\tau=1}^2 \sum_{l=0}^L \sum_{m=-l}^l c_{\tau lm} \mathbf{u}_{\tau lm}(k\mathbf{r}), \quad \mathbf{r} \in \mathbb{R}^3 \setminus \Omega$$

This function satisfies the vector Helmholtz equation and the Silver-Müller radiations conditions. Moreover, its tangential components vanish on $\partial\Omega$. Due to uniqueness of the exterior problem $\mathbf{u} = \mathbf{0}$ everywhere outside Ω . In particular, it vanishes on a sphere, S_R , that circumscribes the region Ω . Orthogonality of the vector spherical harmonics implies

$$\begin{aligned} 0 &= \iint_{S_R} |\mathbf{u}(\mathbf{r})|^2 dS = \sum_{l=0}^L \sum_{m=-l}^l |c_{1\sigma ml}|^2 \left| h_l^{(1)}(kR) \right|^2 \\ &\quad + \sum_{l=0}^L \sum_{m=-l}^l |c_{2\sigma ml}|^2 \left| \frac{(kR h_l^{(1)}(kR))'}{kR} \right|^2 \\ &\quad + \sum_{l=0}^L \sum_{m=-l}^l |c_{2\sigma ml}|^2 l^2 (l+1)^2 \left| \frac{h_l^{(1)}(kR)}{kR} \right|^2 \end{aligned}$$

Each sum must be identically zero and since the spherical Hankel functions, $h_l^{(1)}(z)$ and $(zh_l^{(1)}(z))'$, have no real zeros, see Lemma B.1 on page 77, and we conclude that $c_{\tau lm} = 0$ for all indices. \square

2.5 Expansion of a plane wave

The incident field is commonly taken as a plane wave, which is a well defined vector field everywhere in space. Therefore, the plane wave has an expansion in terms of the regular spherical vector waves $\mathbf{v}_{\tau lm}(k\mathbf{r})$, *i.e.*,

$$\mathbf{E}_i(\mathbf{r}, \omega) = \mathbf{E}_0 e^{ik\hat{\mathbf{k}}_i \cdot \mathbf{r}} = \sum_{\tau=1}^3 \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{\tau lm} \mathbf{v}_{\tau lm}(k\mathbf{r})$$

The expansion coefficients $a_{\tau lm}$ can be computed (see Problem 2.3).

$$\begin{cases} a_{1lm} = 4\pi i^l \mathbf{A}_{1lm}^\dagger(\hat{\mathbf{k}}_i) \cdot \mathbf{E}_0 = 4\pi i^l \mathbf{A}_{2lm}^\dagger(\hat{\mathbf{k}}_i) \cdot (\hat{\mathbf{k}}_i \times \mathbf{E}_0) \\ a_{2lm} = -4\pi i^{l+1} \mathbf{A}_{2lm}^\dagger(\hat{\mathbf{k}}_i) \cdot \mathbf{E}_0 = 4\pi i^{l+1} \mathbf{A}_{1lm}^\dagger(\hat{\mathbf{k}}_i) \cdot (\hat{\mathbf{k}}_i \times \mathbf{E}_0) \\ a_{3lm} = -4\pi i^{l+1} \mathbf{A}_{3lm}^\dagger(\hat{\mathbf{k}}_i) \cdot \mathbf{E}_0 \end{cases} \quad (2.22)$$

The first two are conveniently collected into one expression

$$a_{\tau lm} = 4\pi i^{l+1-\tau} \mathbf{A}_{\tau lm}^\dagger(\hat{\mathbf{k}}_i) \cdot \mathbf{E}_0 = 4\pi i^{l-1+\tau} \mathbf{A}_{\bar{\tau} lm}^\dagger(\hat{\mathbf{k}}_i) \cdot (\hat{\mathbf{k}}_i \times \mathbf{E}_0), \quad \tau = 1, 2 \quad (2.23)$$

The plane wave has to satisfy the Maxwell equations, which implies that the complex vector \mathbf{E}_0 and the direction of the plane wave $\hat{\mathbf{k}}_i$ have to be orthogonal, *i.e.*,

$$\mathbf{E}_0 \cdot \hat{\mathbf{k}}_i = 0$$

Directly, we notice $a_{3n} = 0$ in the expansion, since $\mathbf{A}_{3n}(\hat{\mathbf{k}}_i)$ is proportional to $\hat{\mathbf{k}}_i$, and consequently the summation in the τ -index is only over 1 and 2. An alternative way of expressing this condition is to require that the plane wave has zero divergence. Since only $\tau = 1, 2$ contribute to the sum, the l -summation starts on $l = 1$ ($l = 0$ gives no contribution), *i.e.*,

$$\mathbf{E}_i(\mathbf{r}, \omega) = \mathbf{E}_0 e^{ik\hat{\mathbf{k}}_i \cdot \mathbf{r}} = \sum_n a_n \mathbf{v}_n(k\mathbf{r}) \quad (2.24)$$

In particular, if the incident direction is along the positive z -direction, *i.e.*, $\hat{\mathbf{k}}_i = \hat{\mathbf{z}}$, we get, see (D.9)

$$\begin{cases} a_{1lm} = i^{l+1} \delta_{m\pm 1} \sqrt{(2l+1)\pi} (\hat{\mathbf{x}} \mp i\hat{\mathbf{y}}) \cdot \mathbf{E}_0 \\ a_{2lm} = \pm i^{l+1} \delta_{m\pm 1} \sqrt{(2l+1)\pi} (\hat{\mathbf{x}} \mp i\hat{\mathbf{y}}) \cdot \mathbf{E}_0 \\ a_{3lm} = 0 \end{cases}, \hat{\mathbf{k}}_i = \hat{\mathbf{z}} \quad (2.25)$$

2.6 Far field amplitude

The radiated field has its sources inside the volume V . Outside the circumscribing sphere of the volume V , radius R , the radiated field has an expansion in out-going spherical vector waves, *i.e.*,

$$\mathbf{E}(\mathbf{r}, \omega) = k\sqrt{\eta_0\eta} \sum_{\tau=1}^2 \sum_{l=1}^{\infty} \sum_{m=-l}^l f_{\tau lm} \mathbf{u}_{\tau lm}(k\mathbf{r}) = k\sqrt{\eta_0\eta} \sum_n f_n \mathbf{u}_n(k\mathbf{r}), \quad r > R \quad (2.26)$$

Note that the sum is over $\tau = 1, 2$ and that the l -sum starts at $l = 1$, since only $\tau = 1, 2$ contributes. The corresponding magnetic field has the expansion

$$\mathbf{H}(\mathbf{r}, \omega) = \frac{k}{i\sqrt{\eta_0\eta}} \sum_{\tau=1}^2 \sum_{l=1}^{\infty} \sum_{m=-l}^l f_{\tau lm} \mathbf{u}_{\tau lm}(k\mathbf{r}) = \frac{k}{i\sqrt{\eta_0\eta}} \sum_n f_n \mathbf{u}_{\bar{n}}(k\mathbf{r}), \quad r > R$$

These expansions satisfy the radiation conditions, which are

$$\begin{cases} (\hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r})) - \eta_0\eta \mathbf{H}(\mathbf{r}) = o((kr)^{-1}) \\ \eta_0\eta (\hat{\mathbf{r}} \times \mathbf{H}(\mathbf{r})) + \mathbf{E}(\mathbf{r}) = o((kr)^{-1}) \end{cases} \quad \text{as } r \rightarrow \infty$$

To see this, evaluate the spherical Hankel functions, $h_l^{(1)}$, for large arguments ($kr \gg 1$), see Appendix B.2

$$\begin{cases} h_l^{(1)}(z) = \frac{i^{-l-1}}{z} e^{iz} + O(z^{-2}) \\ \frac{(zh_l^{(1)}(z))'}{z} = \frac{i^{-l}}{z} e^{iz} + O(z^{-2}) \end{cases}$$

and at a large distance from the origin, the out-going spherical vector waves are approximated as

$$\mathbf{u}_{\tau lm}(k\mathbf{r}) = i^{-l-2+\tau} \frac{e^{ikr}}{kr} \mathbf{A}_{\tau lm}(\hat{\mathbf{r}}) + o((kr)^{-1}), \quad \tau = 1, 2$$

Note that this approximation only holds for $\tau = 1, 2$, which are the appropriate τ -index of the electric and the magnetic fields. The fields in the far field zone therefore to leading order are

$$\begin{cases} \mathbf{E}(\mathbf{r}, \omega) = \sqrt{\eta_0 \eta} \frac{e^{ikr}}{r} \sum_n i^{-l-2+\tau} f_n \mathbf{A}_n(\hat{\mathbf{r}}) + o((kr)^{-1}) \\ \mathbf{H}(\mathbf{r}, \omega) = \frac{1}{\sqrt{\eta_0 \eta}} \frac{e^{ikr}}{r} \sum_n i^{-l-\tau} f_n \mathbf{A}_{\bar{n}}(\hat{\mathbf{r}}) + o((kr)^{-1}) \end{cases}$$

We use (2.2), and we get

$$\begin{cases} \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r}, \omega) = \sqrt{\eta_0 \eta} \frac{e^{ikr}}{r} \sum_n i^{-l-\tau} f_n \mathbf{A}_{\bar{n}}(\hat{\mathbf{r}}) + o((kr)^{-1}) \\ \hat{\mathbf{r}} \times \mathbf{H}(\mathbf{r}, \omega) = -\frac{1}{\sqrt{\eta_0 \eta}} \frac{e^{ikr}}{r} \sum_n i^{-l-2+\tau} f_n \mathbf{A}_n(\hat{\mathbf{r}}) + o((kr)^{-1}) \end{cases}$$

From these expressions we see that the radiation conditions are fulfilled, *e.g.*,

$$\begin{aligned} & (\hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r})) - \eta_0 \eta \mathbf{H}(\mathbf{r}) \\ &= \sqrt{\eta_0 \eta} \frac{e^{ikr}}{r} \sum_n i^{-l-\tau} f_n \mathbf{A}_{\bar{n}}(\hat{\mathbf{r}}) - \sqrt{\eta_0 \eta} \frac{e^{ikr}}{r} \sum_n i^{-l-\tau} f_n \mathbf{A}_{\bar{n}}(\hat{\mathbf{r}}) + o((kr)^{-1}) \\ &= o((kr)^{-1}), \quad \text{as } r \rightarrow \infty \end{aligned}$$

and similarly for the other combination.

The general expression of the far field amplitude therefore becomes

$$\mathbf{F}(\hat{\mathbf{r}}) = \sqrt{\eta_0 \eta} \sum_n i^{-l-2+\tau} f_n \mathbf{A}_n(\hat{\mathbf{r}}) \quad (2.27)$$

where

$$\mathbf{E}(\mathbf{r}, \omega) = \frac{e^{ikr}}{r} \mathbf{F}(\hat{\mathbf{r}})$$

For a given far field amplitude $\mathbf{F}(\hat{\mathbf{r}})$, the coefficients f_{1lm} and f_{2lm} are determined by the completeness of the vector spherical harmonics. We get

$$f_{\tau lm} = \frac{i^{l+2-\tau}}{\sqrt{\eta_0 \eta}} \iint_{\Omega} \mathbf{F}(\hat{\mathbf{r}}) \cdot \mathbf{A}_{\tau lm}^{\dagger}(\hat{\mathbf{r}}) \, d\Omega$$

2.6.1 Power transport

The orthogonality relations derived above is useful in the evaluation of the radiation out of a sphere containing the origin and the scatterer. The electric and the magnetic fields are

$$\begin{aligned}\mathbf{E}(\mathbf{r}, \omega) &= k\sqrt{\eta_0\eta} \sum_n (a_n \mathbf{v}_n(k\mathbf{r}) + f_n \mathbf{u}_n(k\mathbf{r})) \\ \mathbf{H}(\mathbf{r}, \omega) &= \frac{1}{i\sqrt{\eta_0\eta}} \sum_n (a_n \nabla \times \mathbf{v}_n(k\mathbf{r}) + f_n \nabla \times \mathbf{u}_n(k\mathbf{r}))\end{aligned}$$

and the radiated power is

$$\begin{aligned}P &= \frac{1}{2} \operatorname{Re} \iint_S \{\mathbf{E} \times \mathbf{H}^*\} \cdot \hat{\nu} \, dS \\ &= -\frac{k}{2} \sum_{nn'} \operatorname{Im} \iint_S \{(a_n \mathbf{v}_n + f_n \mathbf{u}_n) \times (a_{n'}^* \nabla \times \mathbf{v}_{n'}^* + f_{n'}^* \nabla \times \mathbf{u}_{n'}^*)\} \cdot \hat{\nu} \, dS\end{aligned}$$

There are three types of terms in this expression. The first one is, use (2.12)

$$\begin{aligned}2 \operatorname{Im} \sum_{nn'} a_n a_{n'}^* \iint_S \mathbf{v}_n \times (\nabla \times \mathbf{v}_{n'}^*) \cdot \hat{\nu} \, dS \\ = -i \sum_{nn'} a_n a_{n'}^* \iint_S \left(\mathbf{v}_n \times (\nabla \times \mathbf{v}_{n'}^\dagger) - \mathbf{v}_{n'}^\dagger \times (\nabla \times \mathbf{v}_n) \right) \cdot \hat{\nu} \, dS = 0\end{aligned}$$

due to the orthogonality relation (2.15). The second type is analyzed in an analogous manner.

$$\begin{aligned}2 \operatorname{Im} \sum_{nn'} f_n f_{n'}^* \iint_S \mathbf{u}_n \times (\nabla \times \mathbf{u}_{n'}^*) \cdot \hat{\nu} \, dS \\ = -i \sum_{nn'} f_n f_{n'}^* \iint_S \left(\mathbf{u}_n \times (\nabla \times \mathbf{u}_{n'}^\dagger) - \mathbf{u}_{n'}^\dagger \times (\nabla \times \mathbf{u}_n) \right) \cdot \hat{\nu} \, dS = -\frac{2}{k} \sum_n |f_n|^2\end{aligned}$$

using (2.20), since the origin is included. The final term is, using (2.19)

$$\begin{aligned}2 \operatorname{Im} \sum_{nn'} f_n a_{n'}^* \iint_S \mathbf{u}_n \times (\nabla \times \mathbf{v}_{n'}^*) \cdot \hat{\nu} \, dS \\ + 2 \operatorname{Im} \sum_{nn'} a_n f_{n'}^* \iint_S \mathbf{v}_n \times (\nabla \times \mathbf{u}_{n'}^*) \cdot \hat{\nu} \, dS \\ = -i \sum_{nn'} f_n a_{n'}^* \iint_S \left(\mathbf{u}_n \times (\nabla \times \mathbf{v}_{n'}^\dagger) - \mathbf{v}_{n'}^\dagger \times (\nabla \times \mathbf{u}_n) \right) \cdot \hat{\nu} \, dS \\ + i \sum_{nn'} f_n^* a_{n'} \iint_S \left(\mathbf{u}_n^\dagger \times (\nabla \times \mathbf{v}_{n'}) - \mathbf{v}_{n'} \times (\nabla \times \mathbf{u}_n^\dagger) \right) \cdot \hat{\nu} \, dS \\ = -\frac{1}{k} \sum_n (f_n a_n^* + f_n^* a_n) = -\frac{2}{k} \sum_n \operatorname{Re}(f_n a_n^*)\end{aligned}$$

Finally, we get

$$P = \frac{1}{2} \sum_n (|f_n|^2 + \text{Re}(f_n a_n^*)) \quad (2.28)$$

2.7 Expansion of the Green's dyadic

In this section, we develop an expansion of the Green's dyadic of the electric field in free space in terms of spherical vector waves.

2.7.1 Green's dyadic of the electric field in free space

In this section we investigate the electric Green's dyadic \mathbf{G}_e in free space as an expansion in spherical vector waves. We assume the wave number k is a real number. This dyadic satisfies (1.53) on page 28

$$\nabla \times (\nabla \times \mathbf{G}_e(k, |\mathbf{r} - \mathbf{r}'|)) - k^2 \mathbf{G}_e(k, |\mathbf{r} - \mathbf{r}'|) = \mathbf{I} \delta(\mathbf{r} - \mathbf{r}') \quad (2.29)$$

with the appropriate radiation conditions.

For a fixed value of \mathbf{r}' , and each fixed vector \mathbf{a} , $\mathbf{G}_e(k, |\mathbf{r} - \mathbf{r}'|) \cdot \mathbf{a}$ is a function of \mathbf{r} in the domain $r > r'$ and this vector field satisfies the radiation conditions. This function can, of course, be expanded in the vector spherical harmonics on the sphere $r = \text{constant} > r'$. Since the Green's dyadic satisfies (2.29), $\mathbf{G}_e(k, |\mathbf{r} - \mathbf{r}'|) \cdot \mathbf{a}$, as a function of \mathbf{r} for $r > r'$, has an expansion

$$\mathbf{G}_e(k, |\mathbf{r} - \mathbf{r}'|) \cdot \mathbf{a} = \sum_{\substack{n \\ \tau=1,2}} a_n(\mathbf{r}') \mathbf{u}_n(k\mathbf{r}), \quad r > r' \quad (2.30)$$

Notice that only the indices $\tau = 1, 2$ are relevant since $\mathbf{G}_e(k, |\mathbf{r} - \mathbf{r}'|) \cdot \mathbf{a}$ is divergence-free. The expansion coefficients a_n are unknown and have to be determined. This is done by the orthogonality relations in Section 2.3.

Let V be a volume with an inscribed sphere, centered at the origin, that encloses \mathbf{r}' . For all points \mathbf{r} on the bounding surface S we then have $r > r'$. Evaluate the following surface integral, where $\mathbf{A}(\mathbf{r}) = \mathbf{G}_e(k, |\mathbf{r} - \mathbf{r}'|) \cdot \mathbf{a}$ and $\tau' = 1, 2$

$$\begin{aligned} & \iint_S \{ (\hat{\nu} \times \mathbf{v}_n^\dagger) \cdot (\nabla \times \mathbf{A}) + (\hat{\nu} \times (\nabla \times \mathbf{v}_n^\dagger)) \cdot \mathbf{A} \} dS \\ &= \iint_S \{ \mathbf{v}_n^\dagger \times (\nabla \times \mathbf{A}) + (\nabla \times \mathbf{v}_n^\dagger) \times \mathbf{A} \} \cdot \hat{\nu} dS \\ &= \iiint_V \nabla \cdot \{ \mathbf{v}_n^\dagger \times (\nabla \times \mathbf{A}) + (\nabla \times \mathbf{v}_n^\dagger) \times \mathbf{A} \} dv \\ &= \iiint_V \left\{ (\nabla \times \mathbf{v}_n^\dagger) \cdot (\nabla \times \mathbf{A}) - \underbrace{\mathbf{v}_n^\dagger \cdot (\nabla \times (\nabla \times \mathbf{A}))}_{=k^2 \mathbf{A} + \mathbf{a} \delta(\mathbf{r} - \mathbf{r}')} \right. \\ & \quad \left. + \underbrace{(\nabla \times (\nabla \times \mathbf{v}_n^\dagger)) \cdot \mathbf{A}}_{=k^2 \mathbf{v}_n^\dagger} - (\nabla \times \mathbf{v}_n^\dagger) \cdot (\nabla \times \mathbf{A}) \right\} dv \end{aligned}$$

We get

$$\begin{aligned} & \iint_S \{(\hat{\boldsymbol{\nu}} \times \mathbf{v}_n^\dagger) \cdot (\nabla \times \mathbf{A}) + (\hat{\boldsymbol{\nu}} \times (\nabla \times \mathbf{v}_n^\dagger)) \cdot \mathbf{A}\} dS \\ &= - \iiint_V \mathbf{v}_n^\dagger \cdot \mathbf{a} \delta(\mathbf{r} - \mathbf{r}') dv = -\mathbf{v}_n^\dagger(\mathbf{r}') \cdot \mathbf{a} \end{aligned}$$

since $\mathbf{r}' \in V$.

Insert the expansion (2.30), and we get

$$\mathbf{v}_n^\dagger(k\mathbf{r}') \cdot \mathbf{a} = - \sum_{\tau=1,2}^n a_n(\mathbf{r}') \iint_S \{(\hat{\boldsymbol{\nu}} \times \mathbf{v}_{n'}^\dagger) \cdot (\nabla \times \mathbf{u}_n) + (\hat{\boldsymbol{\nu}} \times (\nabla \times \mathbf{v}_{n'}^\dagger)) \cdot \mathbf{u}_n\} dS$$

Use the orthogonality relation (2.19) and we get

$$ik\mathbf{v}_n^\dagger(k\mathbf{r}') \cdot \mathbf{a} = a_n(\mathbf{r}')$$

From (2.30) we have

$$\mathbf{G}_e(k, |\mathbf{r} - \mathbf{r}'|) \cdot \mathbf{a} = ik \sum_{\tau=1,2}^n \mathbf{u}_n(k\mathbf{r}) \mathbf{v}_n^\dagger(k\mathbf{r}') \cdot \mathbf{a}, \quad r > r' \quad (2.31)$$

for all constant vectors \mathbf{a} . This implies that

$$\mathbf{G}_e(k, |\mathbf{r} - \mathbf{r}'|) = ik \sum_{\tau=1,2}^n \mathbf{u}_n(k\mathbf{r}) \mathbf{v}_n^\dagger(k\mathbf{r}'), \quad r > r'$$

Since \mathbf{G}_e is a symmetric dyadic, we have

$$\boxed{\mathbf{G}_e(k, |\mathbf{r} - \mathbf{r}'|) = ik \sum_{\tau=1,2}^n \mathbf{v}_n^\dagger(k\mathbf{r}_<) \mathbf{u}_n(k\mathbf{r}_>) = ik \sum_{\tau=1,2}^n \mathbf{u}_n(k\mathbf{r}_>) \mathbf{v}_n^\dagger(k\mathbf{r}_<), \quad r \neq r'} \quad (2.32)$$

where $\mathbf{r}_<$ ($\mathbf{r}_>$) is the position vector with the smallest (largest) distance to the origin, *i.e.*, if $r < r'$ then $\mathbf{r}_< = \mathbf{r}$ and $\mathbf{r}_> = \mathbf{r}'$. The dagger can be equally well be on the outgoing spherical vector wave by a similar derivation, *i.e.*,

$$\boxed{\mathbf{G}_e(k, |\mathbf{r} - \mathbf{r}'|) = ik \sum_{\tau=1,2}^n \mathbf{v}_n(k\mathbf{r}_<) \mathbf{u}_n^\dagger(k\mathbf{r}_>) = ik \sum_{\tau=1,2}^n \mathbf{u}_n^\dagger(k\mathbf{r}_>) \mathbf{v}_n(k\mathbf{r}_<), \quad r \neq r'}$$

2.7.2 Full Green's dyadic in free space

It is also interest for us to expand the full Green's dyadic \mathbf{G} in free space in spherical vector waves. The definition of the Green's dyadic is

$$\mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) = \mathbf{I}g(k, |\mathbf{r} - \mathbf{r}'|) = \mathbf{I} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

and it is related to the Green's dyadic of the electric field \mathbf{G}_e by (1.54)

$$\mathbf{G} = \mathbf{G}_e - \frac{1}{k^2} \nabla \nabla g$$

An expansion of the scalar Green's function in terms of scalar spherical waves can be obtained using similar derivation as in Section 2.7.1. The result is

$$g(k, |\mathbf{r} - \mathbf{r}'|) = ik \sum_n v_n^\dagger(kr_<) u_n(kr_>) = ik \sum_n v_n(kr_<) u_n^\dagger(kr_>), \quad r \neq r'$$

where

$$\begin{cases} u_n(k\mathbf{r}) = h_l^{(1)}(kr) Y_{lm}(\hat{\mathbf{r}}) \\ v_n(k\mathbf{r}) = j_l(kr) Y_{lm}(\hat{\mathbf{r}}) \end{cases}$$

This result is used to rewrite $\nabla \nabla g$ as

$$\nabla \nabla g(k, |\mathbf{r} - \mathbf{r}'|) = ik \nabla \nabla \sum_n v_n^\dagger(kr_<) u_n(kr_>) = -ik^3 \sum_{\tau=3}^n v_n^\dagger(kr') \mathbf{u}_n(k\mathbf{r}), \quad r \neq r'$$

where we have used the third type of spherical vector waves, $\tau = 3$, defined as

$$\begin{cases} \mathbf{u}_{3lm}(k\mathbf{r}) = \frac{1}{k} \nabla \left(h_l^{(1)}(kr) Y_{lm}(\hat{\mathbf{r}}) \right) \\ \mathbf{v}_{3lm}(k\mathbf{r}) = \frac{1}{k} \nabla \left(j_l(kr) Y_{lm}(\hat{\mathbf{r}}) \right) \end{cases} \quad (2.33)$$

Thus, we have

$$\begin{aligned} \mathbf{G}(k, |\mathbf{r} - \mathbf{r}'|) &= ik \sum_{\tau=1,2,3}^n v_n^\dagger(kr_<) \mathbf{u}_n(kr_>) = ik \sum_{\tau=1,2,3}^n \mathbf{u}_n(kr_>) v_n^\dagger(kr_<) \\ &= ik \sum_{\tau=1,2,3}^n v_n(kr_<) \mathbf{u}_n^\dagger(kr_>) = ik \sum_{\tau=1,2,3}^n \mathbf{u}_n^\dagger(kr_>) v_n(kr_<), \quad r \neq r' \end{aligned}$$

Problems for Chapter 2

2.1 Show that the longitudinal wave, $\tau = 3$, in (2.33)

$$\begin{cases} \mathbf{u}_{3lm}(k\mathbf{r}) = \frac{1}{k} \nabla \left(h_l^{(1)}(kr) Y_{lm}(\hat{\mathbf{r}}) \right) \\ \mathbf{v}_{3lm}(k\mathbf{r}) = \frac{1}{k} \nabla \left(j_l(kr) Y_{lm}(\hat{\mathbf{r}}) \right) \end{cases}$$

also can be written as

$$\begin{cases} \mathbf{u}_{3lm}(k\mathbf{r}) = h_l^{(1)}(kr)' \mathbf{A}_{3lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \frac{h_l^{(1)}(kr)}{kr} \mathbf{A}_{2lm}(\hat{\mathbf{r}}) \\ \mathbf{v}_{3lm}(k\mathbf{r}) = j_l(kr)' \mathbf{A}_{3lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \frac{j_l(kr)}{kr} \mathbf{A}_{2lm}(\hat{\mathbf{r}}) \end{cases}$$

*2.2 Show the following vector identities for the vector spherical harmonics $\mathbf{A}_{\tau lm}$:

$$\left\{ \begin{array}{l} \iint_{\Omega} \mathbf{A}_{1lm}(\hat{\mathbf{r}}) e^{ik\hat{\mathbf{k}}\cdot\mathbf{r}} d\Omega = 4\pi i^l j_l(kr) \mathbf{A}_{1lm}(\hat{\mathbf{k}}) = 4\pi i^l \mathbf{v}_{1lm}(kr) \\ \iint_{\Omega} \mathbf{A}_{2lm}(\hat{\mathbf{r}}) e^{ik\hat{\mathbf{k}}\cdot\mathbf{r}} d\Omega = -\frac{4\pi i^{l+1}}{r} \nabla_k \times \left(j_l(kr) \mathbf{A}_{1lm}(\hat{\mathbf{k}}) \right) = -4\pi i^{l+1} \mathbf{v}_{2lm}(kr) \\ \iint_{\Omega} \mathbf{A}_{3lm}(\hat{\mathbf{r}}) e^{ik\hat{\mathbf{k}}\cdot\mathbf{r}} d\Omega = -\frac{4\pi i^{l+1}}{r} \nabla_k \left(j_l(kr) Y_{lm}(\hat{\mathbf{k}}) \right) = -4\pi i^{l+1} \mathbf{v}_{3lm}(kr) \end{array} \right.$$

where $\mathbf{k} = k\hat{\mathbf{k}}$ and

$$\nabla_k = \hat{\mathbf{x}} \frac{\partial}{\partial k_x} + \hat{\mathbf{y}} \frac{\partial}{\partial k_y} + \hat{\mathbf{z}} \frac{\partial}{\partial k_z}$$

2.3 Show that a plane wave $\mathbf{E}_0 e^{ik\hat{\mathbf{k}}\cdot\mathbf{r}}$ has the following expansion in regular spherical vector waves $\mathbf{v}_{\tau lm}(kr)$

$$\mathbf{E}_0 e^{ik\hat{\mathbf{k}}\cdot\mathbf{r}} = \sum_{l=0}^{\infty} \sum_{m=-l}^l \sum_{\tau=1}^3 a_{\tau lm} \mathbf{v}_{\tau lm}(kr)$$

where

$$\left\{ \begin{array}{l} a_{1lm} = 4\pi i^l \mathbf{A}_{1lm}^\dagger(\hat{\mathbf{k}}) \cdot \mathbf{E}_0 \\ a_{2lm} = -4\pi i^{l+1} \mathbf{A}_{2lm}^\dagger(\hat{\mathbf{k}}) \cdot \mathbf{E}_0 \\ a_{3lm} = -4\pi i^{l+1} \mathbf{A}_{3lm}^\dagger(\hat{\mathbf{k}}) \cdot \mathbf{E}_0 \end{array} \right.$$

Hint: Use the result of Problem 2.2.

Vectors and linear transformations

This appendix contains a short overview of the concept of vectors and linear transformations (dyadics) of vectors, and how these are represented in terms of their components. Transformations between different rotated coordinate systems are also reviewed as well as the corresponding transformation of the components of a dyadic.

A.1 Vectors

In this textbook we denote vectors in italic bold face, *e.g.*, \mathbf{u} . Vectors are in general functions of space and time coordinates, *i.e.*, $\mathbf{u} = \mathbf{u}(\mathbf{r}, t)$, but in this appendix these variables are suppressed since they are not essential for the analysis. A vector that depends on the space and time coordinates is called a *vector field*.

In a particular Cartesian coordinate system $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ the representation of a vector \mathbf{u} is ¹

$$\mathbf{u} = \hat{\mathbf{e}}_1 u_1 + \hat{\mathbf{e}}_2 u_2 + \hat{\mathbf{e}}_3 u_3$$

A “hat” or caret ($\hat{}$) over a vector, *e.g.*, $\hat{\mathbf{e}}_1$, denotes that the vector has unit length (unit vector). The components of the vector, u_i , are obtained by the scalar product

$$u_i = \mathbf{u} \cdot \hat{\mathbf{e}}_i \quad i = 1, 2, 3$$

The components of the vector are often written as a column vector.

$$[\mathbf{u}] = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

In this textbook we use brackets around the vector, $[\mathbf{u}]$, to indicate that we refer to the components of the vector in a particular coordinate system and not the vector \mathbf{u} itself. Notice that the vector \mathbf{u} is geometric quantity, which is independent of

¹We assume the unit vectors (basis vectors) $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ are orthonormal and right-hand oriented.

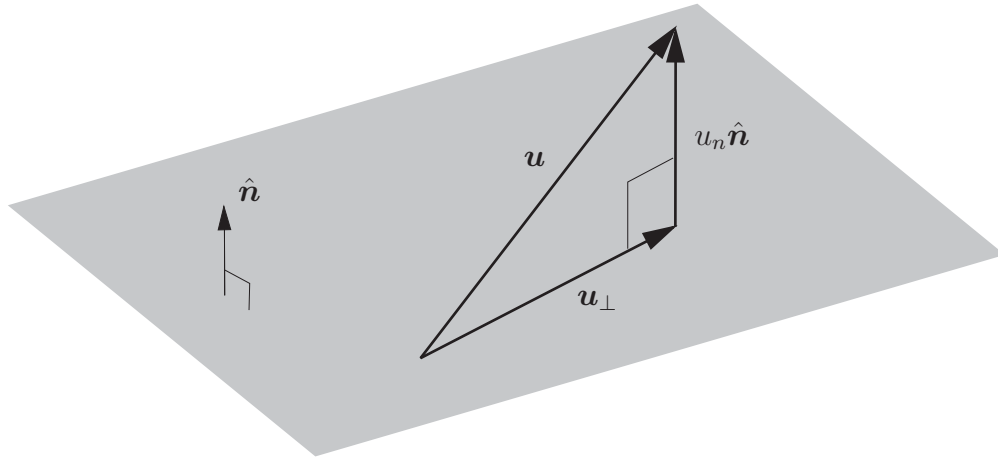


Figure A.1: Projection of a vector \mathbf{u} in a component, $u_n \hat{\mathbf{n}}$, along the direction $\hat{\mathbf{n}}$ and a component, \mathbf{u}_\perp , perpendicular to this direction.

coordinate system. The coordinate representation, $[\mathbf{u}]$, on the other hand, depends on the coordinate system we use to represent the vector in, *i.e.*, the components are different in different coordinate systems, but the vector itself remains the same.

In wave propagation problems, the vectors are decomposed in its components along a given direction (usually the direction of propagation) $\hat{\mathbf{n}}$, and a component lying in the plane orthogonal to the direction $\hat{\mathbf{n}}$. We denote this decomposition or projection of a vector \mathbf{u} as a sum of two parts \mathbf{u}_\perp and u_n , see Figure A.1

$$\mathbf{u} = \mathbf{u}_\perp + u_n \hat{\mathbf{n}}$$

where

$$\begin{cases} u_n = \mathbf{u} \cdot \hat{\mathbf{n}} \\ \mathbf{u}_\perp = \mathbf{u} - \hat{\mathbf{n}}(\mathbf{u} \cdot \hat{\mathbf{n}}) = -\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \hat{\mathbf{u}}) \end{cases} \quad (\text{A.1})$$

by the use of the BAC-CAB rule, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$.

A.2 Linear transformations, matrices and dyadics

Often, we have to deal with linear mappings from one vector field to other vector field, *i.e.*, mapping of a vector \mathbf{u} to another vector \mathbf{v} — both in general functions of the space and time coordinates \mathbf{r} and t . The most simple type of linear transformation is

$$\mathbf{v} = \mathbf{a} \underbrace{(\mathbf{b} \cdot \mathbf{u})}_{\text{scalar}}$$

The vector \mathbf{u} is here mapped to a new vector \mathbf{v} in a new direction along the vector \mathbf{a} . The scaling of the vector is made with the vector \mathbf{b} by the scalar product $\mathbf{b} \cdot \mathbf{u}$. This mapping is called a (*simple*) *dyadic*, and we use the symbol \mathbf{ab} for this

transformation, together with the dyadic product (either with or without parenthesis around the transformation \mathbf{ab}) defined as

$$\mathbf{v} = (\mathbf{ab}) \cdot \mathbf{u} = \mathbf{ab} \cdot \mathbf{u} \stackrel{\text{def}}{=} \mathbf{a} (\mathbf{b} \cdot \mathbf{u})$$

Note that the vectors \mathbf{a} and \mathbf{b} in the transformation \mathbf{ab} are forming a new quantity without any signs between the vectors. The components of the vector \mathbf{v} are

$$v_i = a_i \sum_{j=1}^3 b_j u_j \quad i = 1, 2, 3$$

Simple dyadics can be used as building blocks for general linear transformations of vector fields. The sum of two dyadics, *e.g.*, $\mathbf{A} = \mathbf{a}_1 \mathbf{b}_1 + \mathbf{a}_2 \mathbf{b}_2$, is a new dyadic defined in the following natural way:

$$\mathbf{v} = \mathbf{A} \cdot \mathbf{u} = (\mathbf{a}_1 \mathbf{b}_1 + \mathbf{a}_2 \mathbf{b}_2) \cdot \mathbf{u} = \mathbf{a}_1 \mathbf{b}_1 \cdot \mathbf{u} + \mathbf{a}_2 \mathbf{b}_2 \cdot \mathbf{u}$$

Note that the linear transformation (dyadic) \mathbf{A} is written in roman bold face to distinguish the quantity from the vector \mathbf{A} (italic bold face).

In a specific coordinate system $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ we represent the general linear transformation \mathbf{A} from a vector field \mathbf{u} to another vector field \mathbf{v} by simple dyadics $\hat{\mathbf{e}}_i \hat{\mathbf{e}}_j$, $i, j = 1, 2, 3$.

$$\mathbf{A} = \sum_{i,j=1}^3 A_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \quad (\text{A.2})$$

The action on a vector field \mathbf{u} becomes

$$\mathbf{v} = \mathbf{A} \cdot \mathbf{u} \stackrel{\text{def}}{=} \sum_{i,j=1}^3 \hat{\mathbf{e}}_i A_{ij} (\hat{\mathbf{e}}_j \cdot \mathbf{u}) = \sum_{i,j=1}^3 \hat{\mathbf{e}}_i A_{ij} u_j$$

or in its components

$$v_i = \sum_{j=1}^3 A_{ij} u_j \quad i = 1, 2, 3$$

The component representation of the linear transformation or dyadic \mathbf{A} is represented by a matrix

$$[\mathbf{A}] = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

Just as with vectors, we denote the matrix of the dyadic \mathbf{A} by enclosing the dyadic by brackets, *i.e.*, $[\mathbf{A}]$, to distinguish the coordinate representation of the dyadic from the dyadic \mathbf{A} itself, which is independent of any coordinate representation. This convention is similar to the one we use for a vector \mathbf{u} and its coordinate representation $[\mathbf{u}]$.

By the notion $\mathbf{u}_i = \sum_{j=1}^3 A_{ij} \hat{\mathbf{e}}_j$, $i = 1, 2, 3$, we see that the general linear transformation \mathbf{A} always can be written as

$$\mathbf{A} = \hat{\mathbf{e}}_1 \mathbf{u}_1 + \hat{\mathbf{e}}_2 \mathbf{u}_2 + \hat{\mathbf{e}}_3 \mathbf{u}_3$$

which shows that three dyadics is enough to represent a general linear transformation or dyadic.

So far, the action of vector, \mathbf{u} , on a dyadic, \mathbf{A} , has been from the right by a scalar multiplication. However, some of the advantages with the dyadic notation are seen if we let the action take place from the left. This action leads to the transpose operation. To this end, make a scalar multiplication from the left, that gives a new vector field \mathbf{v} defined by, see (A.2)

$$\mathbf{v} = \mathbf{u} \cdot \mathbf{A} \stackrel{\text{def}}{=} \sum_{i,j=1}^3 (\hat{\mathbf{e}}_i \cdot \mathbf{u}) A_{ij} \hat{\mathbf{e}}_j = \sum_{i,j=1}^3 \hat{\mathbf{e}}_j A_{ij} u_i = \sum_{i=1}^3 u_i \mathbf{u}_i$$

or in component form

$$v_i = \sum_{j=1}^3 A_{ji} u_j \quad i = 1, 2, 3$$

We observe that a scalar multiplication from the left gives the transpose of the matrix $[\mathbf{A}]$. This operation defines the transposed dyadic, \mathbf{A}^t .

$$\mathbf{v} = \mathbf{A}^t \cdot \mathbf{u} \stackrel{\text{def}}{=} \mathbf{u} \cdot \mathbf{A}$$

and

$$[\mathbf{A}^t] = [\mathbf{A}]^t$$

In particular,

$$\mathbf{u}_1 \cdot (\mathbf{A} \cdot \mathbf{u}_2) = (\mathbf{u}_2 \cdot \mathbf{A}^t) \cdot \mathbf{u}_1 = \mathbf{u}_2 \cdot (\mathbf{A}^t \cdot \mathbf{u}_1)$$

for all complex-valued vectors \mathbf{u}_1 and \mathbf{u}_2 .

The inverse dyadic, \mathbf{A}^{-1} , of a dyadic \mathbf{A} can be defined as the matrix inverse of its matrix representation in a specific coordinate system, *i.e.*,

$$[\mathbf{A}^{-1}] = [\mathbf{A}]^{-1} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^{-1}$$

if the coordinate representation of \mathbf{A} is

$$[\mathbf{A}] = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

Similarly, we define the (complex) conjugate of a dyadic, \mathbf{A}^* , and its conjugate transpose or Hermitian transpose, \mathbf{A}^\dagger , of a dyadic \mathbf{A} by the coordinate representations

$$[\mathbf{A}^*] = [\mathbf{A}]^* = \begin{pmatrix} A_{11}^* & A_{12}^* & A_{13}^* \\ A_{21}^* & A_{22}^* & A_{23}^* \\ A_{31}^* & A_{32}^* & A_{33}^* \end{pmatrix} \quad [\mathbf{A}^\dagger] = [\mathbf{A}]^\dagger = \begin{pmatrix} A_{11}^* & A_{21}^* & A_{31}^* \\ A_{12}^* & A_{22}^* & A_{32}^* \\ A_{13}^* & A_{23}^* & A_{33}^* \end{pmatrix}$$

We find that $\mathbf{A}^\dagger = \mathbf{A}^{*t} = \mathbf{A}^{t*}$. A dyadic is called symmetric (Hermitian) if $\mathbf{A}^t = \mathbf{A}$ ($\mathbf{A}^\dagger = \mathbf{A}$). If the dyadic satisfies $\mathbf{A}^t = -\mathbf{A}$ ($\mathbf{A}^\dagger = -\mathbf{A}$), the dyadic is called anti-symmetric (anti-Hermitian). A general dyadic can be decomposed in a symmetric (Hermitian) and an anti-symmetric (anti-Hermitian) part as

$$\mathbf{A} = \mathbf{A}_s + \mathbf{A}_a \quad (\mathbf{A} = \mathbf{A}_H + \mathbf{A}_{a-H})$$

where

$$\begin{cases} \mathbf{A}_s = \frac{1}{2} (\mathbf{A} + \mathbf{A}^t) \\ \mathbf{A}_a = \frac{1}{2} (\mathbf{A} - \mathbf{A}^t) \end{cases} \quad \left(\begin{cases} \mathbf{A}_H = \frac{1}{2} (\mathbf{A} + \mathbf{A}^\dagger) \\ \mathbf{A}_{a-H} = \frac{1}{2} (\mathbf{A} - \mathbf{A}^\dagger) \end{cases} \right)$$

Similarly, a complex-valued dyadic can be decomposed in its real and imaginary parts as

$$\mathbf{A} = \mathbf{A}_r + i\mathbf{A}_i$$

where

$$\begin{cases} \mathbf{A}_r = \frac{1}{2} (\mathbf{A} + \mathbf{A}^\dagger) \\ \mathbf{A}_i = \frac{1}{2i} (\mathbf{A} - \mathbf{A}^\dagger) \end{cases}$$

Notice that both \mathbf{A}_r and \mathbf{A}_i are Hermitian dyadics, and that the off-diagonal elements of \mathbf{A}_r and \mathbf{A}_i in general are not real numbers (the diagonal elements are). For any $\mathbf{u} \in \mathbb{C}^3$

$$\begin{cases} \operatorname{Re} \{ \mathbf{u}^* \cdot \mathbf{A} \cdot \mathbf{u} \} = \mathbf{u}^* \cdot \mathbf{A}_r \cdot \mathbf{u} \\ \operatorname{Im} \{ \mathbf{u}^* \cdot \mathbf{A} \cdot \mathbf{u} \} = \mathbf{u}^* \cdot \mathbf{A}_i \cdot \mathbf{u} \end{cases}$$

Analogously, the vector product between a vector field \mathbf{u} and a dyadic \mathbf{A} is defined.

$$\mathbf{B} = \mathbf{A} \times \mathbf{u} \stackrel{\text{def}}{=} \sum_{i,j=1}^3 \hat{\mathbf{e}}_i A_{ij} (\hat{\mathbf{e}}_j \times \mathbf{u})$$

The vector product can also be applied from the left. We get

$$\mathbf{B} = \mathbf{u} \times \mathbf{A} \stackrel{\text{def}}{=} \sum_{i,j=1}^3 (\mathbf{u} \times \hat{\mathbf{e}}_i) A_{ij} \hat{\mathbf{e}}_j$$

A.2.1 Projections

Projections of a vector \mathbf{u} on a plane with unit normal vector $\hat{\mathbf{n}}$ is a linear mapping with a dyadic, see (A.1)

$$\mathbf{u}_\perp = \mathbf{u} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{u}) = \mathbf{I}_\perp \cdot \mathbf{u}$$

where the projection dyadic \mathbf{I}_\perp is

$$\mathbf{I}_\perp = \mathbf{I}_3 - \hat{\mathbf{n}}\hat{\mathbf{n}}$$

In the same manner, an arbitrary dyadic \mathbf{A} can be decomposed into components parallel and perpendicular to a given fixed direction $\hat{\mathbf{n}}$. We obtain this decomposition by employing $\mathbf{I}_3 = \mathbf{I}_\perp + \hat{\mathbf{n}}\hat{\mathbf{n}}$

$$\mathbf{v} = \mathbf{A} \cdot \mathbf{u} = (\mathbf{I}_\perp + \hat{\mathbf{n}}\hat{\mathbf{n}}) \cdot \mathbf{A} \cdot (\mathbf{I}_\perp + \hat{\mathbf{n}}\hat{\mathbf{n}}) \cdot \mathbf{u}$$

This decomposition implies that every dyadic \mathbf{A} can be written as:

$$\mathbf{A} = \mathbf{A}_{\perp\perp} + \hat{\mathbf{n}}\mathbf{A}_n + \mathbf{A}_\perp\hat{\mathbf{n}} + \hat{\mathbf{n}}A_{nn}\hat{\mathbf{n}} \quad (\text{A.3})$$

where we introduced the notation

$$\begin{cases} \mathbf{A}_{\perp\perp} = \mathbf{I}_\perp \cdot \mathbf{A} \cdot \mathbf{I}_\perp \\ \mathbf{A}_n = \hat{\mathbf{n}} \cdot \mathbf{A} \cdot \mathbf{I}_\perp \\ \mathbf{A}_\perp = \mathbf{I}_\perp \cdot \mathbf{A} \cdot \hat{\mathbf{n}} \\ A_{nn} = \hat{\mathbf{n}} \cdot \mathbf{A} \cdot \hat{\mathbf{n}} \end{cases}$$

Note that the dyadic $\mathbf{A}_{\perp\perp}$ is a two-dimensional dyadic (mapping vectors in the plane with normal $\hat{\mathbf{n}}$ into the same plane). Similarly, the vectors \mathbf{A}_n and \mathbf{A}_\perp are two-dimensional vectors, and A_{nn} is a scalar. If we let the direction $\hat{\mathbf{n}}$ be the unit vector $\hat{\mathbf{e}}_3$, we obtain the coordinate representations of these quantities (we use the index 3 instead of n)

$$[\mathbf{A}_{\perp\perp}] = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [\mathbf{A}_\perp] = \begin{pmatrix} A_{13} \\ A_{23} \\ 0 \end{pmatrix} \quad [\mathbf{A}_3] = \begin{pmatrix} A_{31} \\ A_{32} \\ 0 \end{pmatrix}$$

Such a decomposition is proved useful in the analysis of wave propagation in planar structures.

A.3 Rotation of coordinate system

Several textbooks deal with rotations in space, *e.g.*, the excellent book by Kuipers [8], which also contains the concept of quaternions and its connections to rotations in \mathbb{R}^3 . For details we refer to these textbooks.

Two coordinate systems $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ and $(\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3)$, both orthonormal and right-hand oriented, share the common origin. An example of two such systems is depicted in Figure A.2.

Since the unprimed unit vectors form a basis, each of the primed unit vectors $(\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{e}}'_3)$ can be expressed as a linear combination of the unprimed unit vectors $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$. We have

$$\begin{cases} \hat{\mathbf{e}}'_1 = \hat{\mathbf{e}}_1 a_{11} + \hat{\mathbf{e}}_2 a_{12} + \hat{\mathbf{e}}_3 a_{13} \\ \hat{\mathbf{e}}'_2 = \hat{\mathbf{e}}_1 a_{21} + \hat{\mathbf{e}}_2 a_{22} + \hat{\mathbf{e}}_3 a_{23} \\ \hat{\mathbf{e}}'_3 = \hat{\mathbf{e}}_1 a_{31} + \hat{\mathbf{e}}_2 a_{32} + \hat{\mathbf{e}}_3 a_{33} \end{cases}$$

or

$$\hat{\mathbf{e}}'_i = \sum_{j=1}^3 \hat{\mathbf{e}}_j a_{ij} \quad i = 1, 2, 3$$

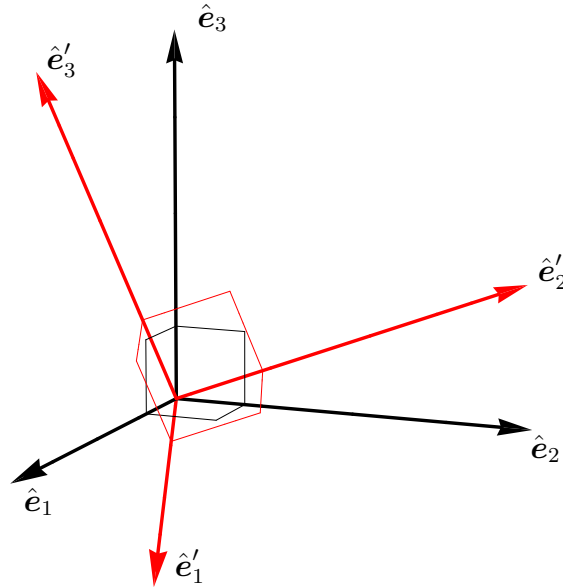


Figure A.2: Two rotated coordinate systems $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ and $(\hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$ with a common origin. The original is shown in black and the rotated in red.

Since we have assumed that the unit vectors $(\hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$ and $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ both are right-hand oriented, the determinant of the matrix $[\mathbf{A}]$, with entries a_{ij} , is 1, *i.e.*, $\det[\mathbf{A}] = 1$.

The components a_{ij} , $i, j = 1, 2, 3$, are the direction cosines² between the axis i and j .

$$a_{ij} = \hat{e}'_i \cdot \hat{e}_j \quad i, j = 1, 2, 3$$

In general, we have

$$\hat{e}'_i \cdot \hat{e}_j = a_{ij} \neq a_{ji} = \hat{e}'_j \cdot \hat{e}_i \quad i, j = 1, 2, 3$$

Similarly, the unprimed unit vectors \hat{e}_i can be written as a linear combination of the primed ones $(\hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$. We get

$$\hat{e}_i = \hat{e}'_1 (\hat{e}'_1 \cdot \hat{e}_i) + \hat{e}'_2 (\hat{e}'_2 \cdot \hat{e}_i) + \hat{e}'_3 (\hat{e}'_3 \cdot \hat{e}_i) \quad i = 1, 2, 3$$

or expressed in the direction cosines a_{ij}

$$\begin{cases} \hat{e}_1 = \hat{e}'_1 a_{11} + \hat{e}'_2 a_{21} + \hat{e}'_3 a_{31} \\ \hat{e}_2 = \hat{e}'_1 a_{12} + \hat{e}'_2 a_{22} + \hat{e}'_3 a_{32} \\ \hat{e}_3 = \hat{e}'_1 a_{13} + \hat{e}'_2 a_{23} + \hat{e}'_3 a_{33} \end{cases}$$

In short

$$\hat{e}_i = \sum_{j=1}^3 \hat{e}'_j a_{ji} \quad i = 1, 2, 3$$

²Also given as $a_{ij} = \cos(x_i, x_j)$ where (x_i, x_j) is the angle between \hat{e}'_i and \hat{e}_j axes.

From these expressions we see that if the transformation $\hat{e}_i \rightarrow \hat{e}'_i$ is made with a_{ij} then the transformation $\hat{e}'_i \rightarrow \hat{e}_i$ is made with a_{ji} . The matrix $[\mathbf{A}]$ therefore is an orthogonal matrix, *i.e.*, $[\mathbf{A}]^{-1} = [\mathbf{A}]^t$.

We are now ready to give the formal definition of a vector. A vector \mathbf{u} is a geometric quantity with components (u_1, u_2, u_3) in the system $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$, which are related to the components (u'_1, u'_2, u'_3) in the system $(\hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$ by the direction cosines, a_{ij} , $i, j = 1, 2, 3$, in the following way:

$$u'_i = \sum_{j=1}^3 a_{ij} u_j \quad i = 1, 2, 3 \quad (\text{A.4})$$

or expressed as column vectors and standard matrix multiplication

$$\begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Definition A.1. A physical quantity \mathbf{u} which components u_i and u'_i in two rotated coordinate systems, respectively, are related to each other by (A.4) is called a **vector**³.

This definition implies that for every vector \mathbf{u} we have

$$\begin{aligned} \mathbf{u} &= \hat{e}'_1 u'_1 + \hat{e}'_2 u'_2 + \hat{e}'_3 u'_3 = \sum_{i=1}^3 u'_i \hat{e}'_i \\ &= \sum_{i,j=1}^3 a_{ij} u_j \hat{e}'_i = \sum_{j=1}^3 u_j \hat{e}_j = \hat{e}_1 u_1 + \hat{e}_2 u_2 + \hat{e}_3 u_3 \end{aligned}$$

By this definition every vector becomes a quantity which is independent of the coordinate representation, just as we required in the beginning of this appendix.

In a similar way, we define a dyadic (or a tensor of the second kind).

Definition A.2. A physical quantity \mathbf{D} which components D_{ij} and D'_{ij} in two rotated coordinate systems, respectively, are related by

$$D'_{ij} = \sum_{k,l=1}^3 a_{ik} a_{jl} D_{kl} \quad i, j = 1, 2, 3$$

is called a **dyadic**. This relation can also be expressed as similarity transformation

$$[\mathbf{D}]' = [\mathbf{A}] [\mathbf{D}] [\mathbf{A}]^t \quad (\text{A.5})$$

³A vector is also called a polar vector which distinguishes it from an axial vector which transforms by (A.4) where $\det[\mathbf{A}] = -1$. If the vector depends on the space coordinates, *i.e.*, \mathbf{u} is a vector field, then also the space coordinates are transformed as

$$u'_i(x'_1, x'_2, x'_3) = \sum_{j=1}^3 a_{ij} u_j(x_1, x_2, x_3) \quad i = 1, 2, 3$$

where x'_1, x'_2, x'_3 and x_1, x_2, x_3 are the components of the position vector in the primed and the unprimed systems, respectively.

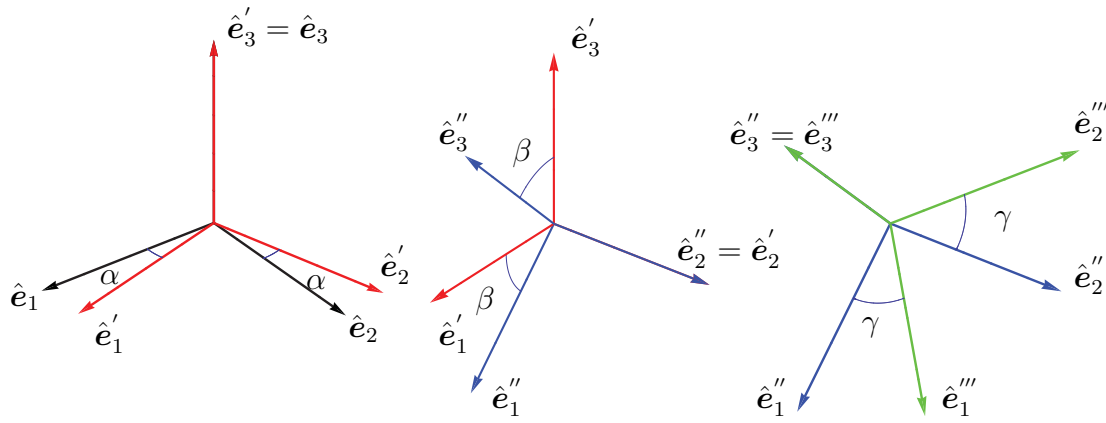


Figure A.3: Definition of the three Euler angles α , β and γ . The first rotation has red coordinate axes, the second rotation blue ones, and the final rotation has green coordinate axes.

A.3.1 Euler angles

The two rotated coordinate systems, $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ and $(\hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$, are related to each other by the direction cosines. These relations can also be expressed in an alternative way by the three Euler angles α , β , γ . These angles are defined by three consecutive rotations, see Figure A.3. The three rotations are explicitly given by:

1. A rotation with the angle α around the \hat{e}_3 axis
2. A rotation with the angle β around the \hat{e}'_2 axis
3. A rotation with the angle γ around the \hat{e}''_3 axis

The three different rotations are represented by the following matrices:

1. The first rotation is represented by

$$[\mathbf{R}_1] = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. The second rotation is represented by

$$[\mathbf{R}_2] = \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix}$$

3. The third rotation is represented by

$$[\mathbf{R}_3] = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In total, the rotation is made by

$$[\mathbf{A}] = [\mathbf{R}_3][\mathbf{R}_2][\mathbf{R}_1]$$

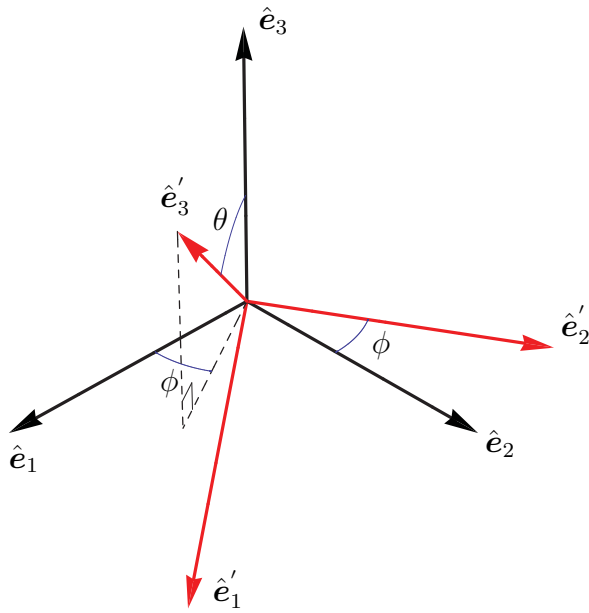


Figure A.4: The angles of rotation θ and ϕ . The unit vector \hat{e}'_2 lies in the \hat{e}_1 - \hat{e}_2 -plane in this figure.

In this textbook, we are often using similarity transformations of linear transformations (dyadics). These are made by a *rotation matrix* $[\mathbf{R}]$ consisting of a combination of two rotations. The spherical angles θ and ϕ are defined in Figure A.4. The relation between these angles and the Euler angles α , β and γ is:

$$\alpha = \phi, \quad \beta = \theta, \quad \gamma = 0$$

We get

$$\begin{aligned} [\mathbf{R}] &= [\mathbf{R}_2][\mathbf{R}_1] = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ \sin \phi & -\cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix} \end{aligned} \quad (\text{A.6})$$

A.3.2 Quaternions

Quaternions are useful in representing rotations in \mathbb{R}^3 , and they are an extension of the complex numbers. An excellent introduction of the quaternions are found in [8], and here we only give a short summary of their use.

Introduce the hyper-complex numbers of rank 4, which also is called the quaternions. These numbers were introduced 1843 by Hamilton⁴ to generalize the complex numbers.

⁴Sir William Rowan Hamilton (1805–1865), Irish physicist, astronomer, and mathematician.

We start by introducing an orthonormal basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$, in \mathbb{R}^3 together with the following multiplication algebra:

$$\begin{cases} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1 \\ \mathbf{ij} = \mathbf{k} = -\mathbf{ji} \\ \mathbf{jk} = \mathbf{i} = -\mathbf{kj} \\ \mathbf{ki} = \mathbf{j} = -\mathbf{ik} \end{cases} \quad (\text{A.7})$$

Notice the similarities with the multiplication rule for complex numbers $i^2 = -1$.

A quaternion q is a 4-tuple of real numbers, *i.e.*,

$$q = (q_0, q_1, q_2, q_3)$$

The first entry q_0 is called the scalar part of the quaternion and the remaining three are called the vector part of the quaternion. We adopt the notation

$$\mathbf{q} = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$$

and we write

$$q = q_0 + \mathbf{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$$

We see that it is possible to identify real vectors in \mathbb{R}^3 with quaternions with zero scalar parts. Such quaternions are called pure quaternions, *i.e.*,

$$\mathbf{q} \in \mathbb{R}^3 \quad \Leftrightarrow \quad q = 0 + \mathbf{q} \text{ (pure quaternion)}$$

Let q and p be two quaternions, *i.e.*,

$$q = q_0 + \mathbf{q}, \quad p = p_0 + \mathbf{p}$$

The quaternions are equal provided $q_i = p_i$, $i = 0, 1, 2, 3$.

We define addition of two quaternions as

$$q + p = p + q = q_0 + p_0 + \mathbf{p} + \mathbf{q}$$

Multiplication of two quaternions is more complex, and we use the product rules in (A.7).

$$\begin{aligned} pq &= (p_0 + \mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3)(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3) \\ &= p_0(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3) + \mathbf{i}p_1(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3) \\ &\quad + \mathbf{j}p_2(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3) + \mathbf{k}p_3(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3) \end{aligned}$$

The scalar part of this product is

$$p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 = p_0q_0 - \mathbf{p} \cdot \mathbf{q}$$

The vector part of this product becomes

$$\begin{aligned} &p_0(\mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3) + p_1(\mathbf{i}q_0 + \mathbf{k}q_2 - \mathbf{j}q_3) \\ &\quad + p_2(\mathbf{j}q_0 - \mathbf{k}q_1 + \mathbf{i}q_3) + p_3(\mathbf{k}q_0 + \mathbf{j}q_1 - \mathbf{i}q_2) \\ &= p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{i}(p_2q_3 - p_3q_2) + \mathbf{j}(p_3q_1 - p_1q_3) + \mathbf{k}(p_1q_2 - p_2q_1) \\ &= p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q} \end{aligned}$$

The product pq therefore becomes

$$pq = p_0q_0 - \mathbf{p} \cdot \mathbf{q} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q} \quad (\text{A.8})$$

In particular, the product of a quaternion q and a pure quaternion \mathbf{v} (vector in \mathbb{R}^3) is

$$q\mathbf{v} = (q_0 + \mathbf{q})(0 + \mathbf{v}) = -\mathbf{q} \cdot \mathbf{v} + q_0\mathbf{v} + \mathbf{q} \times \mathbf{v} \quad (\text{A.9})$$

Two more concepts are useful. The complex conjugate of a quaternion $q = q_0 + \mathbf{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ is denoted q^* and defined as

$$q^* = q_0 - \mathbf{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3$$

From (A.8) it is easy to prove that

$$(pq)^* = q^*p^*$$

Moreover, the norm of a quaternion $q = q_0 + \mathbf{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ is denoted $|q|$ and defined as

$$|q| = \sqrt{q^*q}$$

This definition makes sense since, using the product rule (A.8), we have

$$q^*q = (q_0 - \mathbf{q})(q_0 + \mathbf{q}) = q_0^2 + \mathbf{q} \cdot \mathbf{q} + q_0\mathbf{q} - q_0\mathbf{q} - \mathbf{q} \times \mathbf{q} = q_0^2 + |\mathbf{q}|^2$$

which always is a non-negative real number, and

$$qq^* = q_0^2 + |\mathbf{q}|^2$$

The inverse of a quaternion q is denoted q^{-1} and satisfies

$$qq^{-1} = q^{-1}q = 1$$

Multiply from the left or right with q^*

$$\underbrace{q^*q}_{|q|^2} q^{-1} = q^* \quad q^{-1} \underbrace{qq^*}_{|q|^2} = q^*$$

This implies

$$q^{-1} = \frac{q^*}{|q|^2}$$

Notice the similarities with the complex numbers $z^{-1} = z^*/|z|^2$

To connect to a rotation of a vector \mathbf{v} in \mathbb{R}^3 , we evaluate

$$\begin{aligned} q\mathbf{v}q^* &= (-\mathbf{q} \cdot \mathbf{v} + q_0\mathbf{v} + \mathbf{q} \times \mathbf{v})(q_0 - \mathbf{q}) \\ &= -q_0\mathbf{q} \cdot \mathbf{v} + (q_0\mathbf{v} + \mathbf{q} \times \mathbf{v}) \cdot \mathbf{q} + (\mathbf{q} \cdot \mathbf{v})\mathbf{q} \\ &\quad + q_0(q_0\mathbf{v} + \mathbf{q} \times \mathbf{v}) - (q_0\mathbf{v} + \mathbf{q} \times \mathbf{v}) \times \mathbf{q} \\ &= 2(\mathbf{q} \cdot \mathbf{v})\mathbf{q} + q_0(q_0\mathbf{v} + 2\mathbf{q} \times \mathbf{v}) - |\mathbf{q}|^2\mathbf{v} \end{aligned}$$

where we have used (A.9) and the product rule (A.8). Notice that qvq^* is a pure quaternion and therefore is associated with a vector \mathbf{v}' , *i.e.*,

$$\mathbf{v}' = (q_0^2 - |\mathbf{q}|^2)\mathbf{v} + 2(\mathbf{q} \cdot \mathbf{v})\mathbf{q} + 2q_0\mathbf{q} \times \mathbf{v}$$

We conclude that $L_q(\mathbf{v}) = qvq^*$ is a linear mapping on \mathbb{R}^3 .

We specialize to $q = \cos(\theta/2) + \hat{\mathbf{e}} \sin(\theta/2)$, where $\theta \in [-\pi, \pi]$ and $\hat{\mathbf{e}}$ is a real unit vector. The result is

$$\mathbf{v}' = \cos \theta \mathbf{v} + 2\hat{\mathbf{e}} \sin^2(\theta/2)(\hat{\mathbf{e}} \cdot \mathbf{v}) + \sin \theta \hat{\mathbf{e}} \times \mathbf{v}$$

The norm of this vector is

$$\begin{aligned} |\mathbf{v}'|^2 &= \cos^2 \theta |\mathbf{v}|^2 + 4 \underbrace{(\cos \theta + \sin^2(\theta/2))}_{(1+\cos \theta)/2 = \cos^2(\theta/2)} \sin^2(\theta/2) (\hat{\mathbf{e}} \cdot \mathbf{v})^2 + \sin^2 \theta \underbrace{|\hat{\mathbf{e}} \times \mathbf{v}|^2}_{|\mathbf{v}|^2 - (\hat{\mathbf{e}} \cdot \mathbf{v})^2} \\ &= |\mathbf{v}|^2 + \sin^2 \theta (\hat{\mathbf{e}} \cdot \mathbf{v})^2 - \sin^2 \theta (\hat{\mathbf{e}} \cdot \mathbf{v})^2 = |\mathbf{v}|^2 \end{aligned}$$

and if we let $\mathbf{v} = \hat{\mathbf{e}}$ we obtain

$$q\hat{\mathbf{e}}q^* = 2\hat{\mathbf{e}} \sin^2(\theta/2) + \underbrace{\cos \theta}_{\cos^2(\theta/2) - \sin^2(\theta/2)} \hat{\mathbf{e}} = \hat{\mathbf{e}}$$

We suspect that the mapping qvq^* with $q = \cos(\theta/2) + \hat{\mathbf{e}} \sin(\theta/2)$ represents a rotation of the vector \mathbf{v} to \mathbf{v}' with rotation axis $\hat{\mathbf{e}}$. We now prove that this is the case, and determine also how large the rotation angle is. To this end, decompose the vector $\mathbf{v} = (\hat{\mathbf{e}} \cdot \mathbf{v})\hat{\mathbf{e}} + \mathbf{u}$, where $\mathbf{u} = \mathbf{v} - (\hat{\mathbf{e}} \cdot \mathbf{v})\hat{\mathbf{e}}$ is orthogonal to $\hat{\mathbf{e}}$, *i.e.*, $\mathbf{u} \cdot \hat{\mathbf{e}} = 0$. We get

$$\mathbf{v}' = L_q((\hat{\mathbf{e}} \cdot \mathbf{v})\hat{\mathbf{e}}) + L_q(\mathbf{u}) = (\hat{\mathbf{e}} \cdot \mathbf{v})\hat{\mathbf{e}} + \underbrace{\cos \theta \mathbf{u} + \sin \theta \hat{\mathbf{e}} \times \mathbf{u}}_{\mathbf{u}'}$$

since $L_q(\hat{\mathbf{e}}) = \hat{\mathbf{e}}$. This proves that the rotation is θ in the positive $\hat{\mathbf{e}}$ -direction, see Figures A.5 and A.6.

Example A.1

Let $\hat{\mathbf{e}} = (1, 1, 1)/\sqrt{3}$ be the axis of rotation and rotate the angle $\theta = \pi/3$. Then the appropriate quaternion is

$$q = \cos(\theta/2) + \hat{\mathbf{e}} \sin(\theta/2) = \frac{1}{2} + \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$$

The rotation of the vector $\mathbf{v} = \mathbf{i}$ then is

$$\begin{aligned} L_q(\mathbf{i}) &= qi q^* = \frac{1}{4}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k})\mathbf{i}(1 - \mathbf{i} - \mathbf{j} - \mathbf{k}) = \frac{1}{4}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k})(\mathbf{i} + 1 - \mathbf{k} + \mathbf{j}) \\ &= \frac{1}{4}(\mathbf{i} + 1 - \mathbf{k} + \mathbf{j} - 1 + \mathbf{i} + \mathbf{j} + \mathbf{k} - \mathbf{k} + \mathbf{j} - \mathbf{i} - 1 + \mathbf{j} + \mathbf{k} + 1 - \mathbf{i}) = \mathbf{j} \end{aligned}$$

■

Two consecutive rotations with angles α and β around a common axis $\hat{\mathbf{e}}$ is a single rotation around $\hat{\mathbf{e}}$ with angle $\alpha + \beta$. In fact, the appropriate quaternions p and q are

$$p = \cos \alpha + \hat{\mathbf{e}} \sin \alpha, \quad q = \cos \beta + \hat{\mathbf{e}} \sin \beta$$

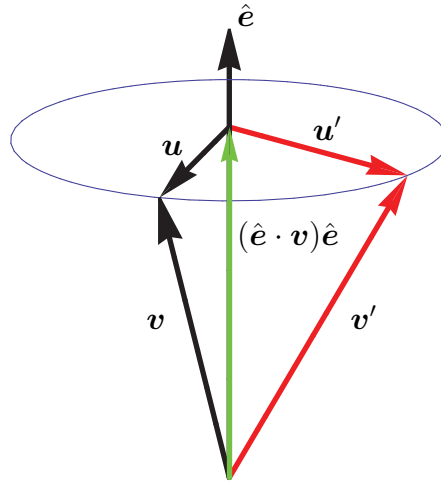


Figure A.5: The decomposition of the vector \mathbf{v} .

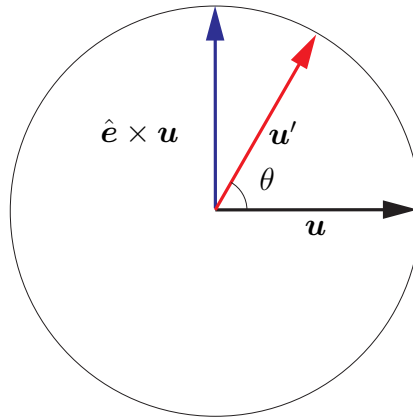


Figure A.6: The plane of rotation.

and the product is

$$\begin{aligned}
 pq &= p_0q_0 - \mathbf{p} \cdot \mathbf{q} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p} \times \mathbf{q} \\
 &= \cos \alpha \cos \beta - \sin \alpha \sin \beta + (\cos \alpha \sin \beta + \cos \beta \sin \alpha)\hat{\mathbf{e}} \\
 &= \cos(\alpha + \beta) + \hat{\mathbf{e}} \sin(\alpha + \beta)
 \end{aligned}$$

which proves the statement.

This observation generalizes to two arbitrary rotations as seen from the following analysis

$$L_p(\mathbf{w}) = p\mathbf{w}p^*, \quad L_q(\mathbf{v}) = q\mathbf{v}q^*$$

The product then is a single rotation quantified by the quaternion pq , *i. e.*,

$$L_p(L_q(\mathbf{v})) = pL_q(\mathbf{v})p^* = pq\mathbf{v}q^*p^* = pq\mathbf{v}(pq)^* = L_{pq}(\mathbf{v})$$

Now when we know that two consecutive rotations can be expressed as a single rotation, the appropriate question is: What is the common axis $\hat{\mathbf{e}}$ and angle of

rotation θ for a product of two consecutive rotations, axes \mathbf{a} and \mathbf{b} and angles α and β ? We have⁵

$$\begin{aligned}\cos(\theta/2) + \hat{\mathbf{e}} \sin(\theta/2) &= q_{\hat{\mathbf{e}},\theta} = q_{\hat{\mathbf{b}},\beta} q_{\hat{\mathbf{a}},\alpha} \\ &= (\cos(\beta/2) + \hat{\mathbf{b}} \sin(\beta/2))(\cos(\alpha/2) + \hat{\mathbf{a}} \sin(\alpha/2)) \\ &= \cos(\beta/2) \cos(\alpha/2) - \hat{\mathbf{b}} \cdot \hat{\mathbf{a}} \sin(\beta/2) \sin(\alpha/2) \\ &\quad + \hat{\mathbf{a}} \cos(\beta/2) \sin(\alpha/2) + \hat{\mathbf{b}} \sin(\beta/2) \cos(\alpha/2) \\ &\quad + \sin(\beta/2) \sin(\alpha/2) \hat{\mathbf{b}} \times \hat{\mathbf{a}}\end{aligned}$$

Identify the scalar and vector parts:

$$\theta = 2 \arccos \left(\cos(\beta/2) \cos(\alpha/2) - \hat{\mathbf{b}} \cdot \hat{\mathbf{a}} \sin(\beta/2) \sin(\alpha/2) \right)$$

and

$$\begin{aligned}\hat{\mathbf{e}} &= \left\{ \hat{\mathbf{a}} \cos(\beta/2) \sin(\alpha/2) + \hat{\mathbf{b}} \sin(\beta/2) \cos(\alpha/2) \right. \\ &\quad \left. + \sin(\beta/2) \sin(\alpha/2) \hat{\mathbf{b}} \times \hat{\mathbf{a}} \right\} / \sin(\theta/2)\end{aligned}$$

These two equations quantifies the angle of rotation and the axis of rotation uniquely.

In the beginning of Section A.3, we showed that rotations can be quantified as a 3×3 orthogonal matrix $[\mathbf{R}]$. We have in this section learned that the same rotation can be quantified by a quaternion q . In fact, any orthogonal matrix $[\mathbf{R}]$ can be identified with a quaternion q , and vice versa. The result is

$$[\mathbf{R}] = \begin{pmatrix} 2(q_0^2 + q_1^2) - 1 & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & 2(q_0^2 + q_2^2) - 1 & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & 2(q_0^2 + q_3^2) - 1 \end{pmatrix}$$

and for the inverse

$$\begin{cases} q_0 = \frac{\text{Tr}(\mathbf{R}) + 1}{2} \\ q_1 = \frac{R_{23} - R_{32}}{4q_0} \\ q_2 = \frac{R_{31} - R_{13}}{4q_0} \\ q_3 = \frac{R_{12} - R_{21}}{4q_0} \end{cases}$$

⁵To clarify the axis of rotation and angle of rotation, these data are added to the quaternion as a subscript.

Bessel functions

In wave propagation problems the Bessel differential equation often appears, especially in problems showing axial or spherical symmetries. This appendix collects some useful and important results for the solution of the Bessel differential equation. Moreover, the modified Bessel functions and the spherical Bessel and Hankel functions are presented. Additional results and some of the derivations are found in *e.g.*, Ref. 1.

B.1 Bessel and Hankel functions

The Bessel differential equation is

$$z^2 \frac{d^2}{dz^2} Z_n(z) + z \frac{d}{dz} Z_n(z) + (z^2 - n^2) Z_n(z) = 0 \quad (\text{B.1})$$

where n is assumed integer¹.

There exist two linearly independent solutions to this differential equation. One is regular at the origin, $z = 0$, and this solution is the Bessel function $J_n(z)$ of order n . The argument z is a complex number. These solutions are often called cylindrical Bessel function of order n , which stresses the affinity to problems with the axial symmetry. The Bessel functions $J_n(z)$ are defined real-valued for a real argument z . An everywhere in the complex z -plane convergent power series is

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{z}{2}\right)^{n+2k} \quad (\text{B.2})$$

We notice immediately that $J_n(z)$ is an even function for even n and odd for odd n , *i.e.*,

$$J_n(-z) = (-1)^n J_n(z)$$

¹A more general definition with *e.g.*, complex-valued n is also possible, but the expressions and the results often differ.

Order n	Root j			
	1	2	3	4
0	2.405	5.520	8.654	11.79
1	3.832	7.016	10.17	13.32
2	5.136	8.417	11.62	14.80
3	6.380	9.761	13.01	16.22
4	7.588	11.06	14.37	17.62

Table B.1: Table of the roots ξ_{nj} to $J_n(z)$.

A commonly used integral representation of the Bessel functions is

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin t - nt) dt = \frac{1}{2\pi} \int_0^{2\pi} e^{iz \cos t} e^{in(t-\frac{1}{2}\pi)} dt \quad (\text{B.3})$$

From this integral representation, we see that the Bessel functions for positive and negative *integer* orders, n , are related to each other.

$$J_{-n}(z) = (-1)^n J_n(z)$$

The power series representation in (B.2) implies that for small arguments we have

$$J_n(z) = \frac{1}{n!} \left(\frac{z}{2}\right)^n + O(z^{n+2})$$

For large arguments hold ($-\pi < \arg z < \pi$)

$$J_n(z) = \left(\frac{2}{\pi z}\right)^{1/2} \left\{ P_n(z) \cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) - Q_n(z) \sin\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) \right\}$$

where the functions $P_n(z)$ and $Q_n(z)$ have the following asymptotic expansions ($\nu = 4n^2$)

$$\begin{cases} P_n(z) \sim 1 - \frac{(\nu-1)(\nu-9)}{2!(8z)^2} + \frac{(\nu-1)(\nu-9)(\nu-25)(\nu-49)}{4!(8z)^4} - \dots \\ Q_n(z) \sim \frac{\nu-1}{8z} - \frac{(\nu-1)(\nu-9)(\nu-25)}{3!(8z)^3} + \dots \end{cases} \quad (\text{B.4})$$

The roots of the Bessel function $J_n(z)$ are all real, and the first roots, ξ_{nj} , are listed in Table B.1. The derivative of the Bessel function $J_n(z)$ has also only real roots, η_{nj} , and the first ones are listed in Table B.2. Larger roots (larger j values) are asymptotically given by

$$\xi_{nj} = j\pi + \left(n - \frac{1}{2}\right) \frac{\pi}{2}, \quad \eta_{nj} = j\pi + \left(n - \frac{3}{2}\right) \frac{\pi}{2}$$

Order n	Root j			
	1	2	3	4
0	3.832	7.016	10.17	13.32
1	1.841	5.331	8.536	11.71
2	3.054	6.706	9.970	13.17
3	4.201	8.015	11.34	14.59
4	5.317	9.282	12.68	15.96

Table B.2: Table of the roots η_{nj} to $J'_n(z)$.

Another, linearly independent solution to the Bessel differential equation, which is real-valued for real arguments, is the Neumann function² $N_n(z)$. The power series expansion is

$$\begin{aligned}
 N_n(z) &= \frac{2}{\pi} \left(\ln \left(\frac{z}{2} \right) + \gamma - \frac{1}{2} \sum_{k=1}^n \frac{1}{k} \right) J_n(z) \\
 &\quad - \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{z}{2} \right)^{n+2k}}{k!(n+k)!} \sum_{l=1}^k \left(\frac{1}{l} + \frac{1}{l+n} \right) \\
 &\quad - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{z}{2} \right)^{-n+2k}
 \end{aligned}$$

where the Euler constant $\gamma = 0.577\,215\,66\dots$, and where all sums are defined as zero if the summation index exceeds the upper summation limit. This solution is singular at the origin $z = 0$. For small arguments the dominant contribution is

$$\begin{aligned}
 N_0(z) &= \frac{2}{\pi} \left(\ln \left(\frac{z}{2} \right) + \gamma \right) (1 + O(z^2)) \\
 N_n(z) &= -\frac{(n-1)!}{\pi} \left(\frac{z}{2} \right)^{-n} (1 + O(z^2)), \quad n > 0
 \end{aligned}$$

For large arguments the Neumann function has an asymptotic expansion ($-\pi < \arg z < \pi$)

$$N_n(z) = \left(\frac{2}{\pi z} \right)^{1/2} \left(P_n(z) \sin \left(z - \frac{n\pi}{2} - \frac{\pi}{4} \right) + Q_n(z) \cos \left(z - \frac{n\pi}{2} - \frac{\pi}{4} \right) \right)$$

where the functions $P_n(z)$ and $Q_n(z)$ are given by (B.4).

In the solution of scattering problems, linear combinations of Bessel and Neumann functions, *i.e.*, the Hankel functions, $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ of the first and the second kind, respectively, are natural³. These are defined as

$$\begin{aligned}
 H_n^{(1)}(z) &= J_n(z) + iN_n(z) \\
 H_n^{(2)}(z) &= J_n(z) - iN_n(z)
 \end{aligned}$$

²These solutions are also called Bessel functions of the second kind.

³These also called Bessel functions of the third kind.

The Hankel functions of the first and second kind have integral representations

$$H_n^{(1)}(z) = \frac{2}{i\pi} e^{-in\frac{\pi}{2}} \int_0^\infty e^{iz \cosh s} \cosh ns \, ds, \quad 0 < \arg z < \pi$$

$$H_n^{(2)}(z) = \frac{2i}{\pi} e^{in\frac{\pi}{2}} \int_0^\infty e^{-iz \cosh s} \cosh ns \, ds, \quad -\pi < \arg z < 0$$

For large arguments, the Hankel functions have asymptotic expansions

$$H_n^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z - \frac{n\pi}{2} - \frac{\pi}{4})} (P_n(z) + iQ_n(z)), \quad -\pi < \arg z < 2\pi$$

$$H_n^{(2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{-i(z - \frac{n\pi}{2} - \frac{\pi}{4})} (P_n(z) - iQ_n(z)), \quad -2\pi < \arg z < \pi$$
(B.5)

where the functions $P_n(z)$ and $Q_n(z)$ are given by (B.4).

Solutions to the Bessel differential equation of different order are related to each other by recursion relations. Some of the more important ones are ($n = 0, 1, 2, \dots, m = 0, 1, 2, \dots$)⁴

$$Z_{n-1}(z) - Z_{n+1}(z) = 2Z'_n(z)$$

$$Z_{n-1}(z) + Z_{n+1}(z) = \frac{2n}{z} Z_n(z)$$

$$Z_{n+1}(z) = \frac{n}{z} Z_n(z) - Z'_n(z)$$

$$Z'_n(z) = Z_{n-1}(z) - \frac{n}{z} Z_n(z)$$

$$\left(\frac{d}{z dz}\right)^m [z^n Z_n(z)] = z^{n-m} Z_{n-m}(z)$$

$$\left(\frac{d}{z dz}\right)^m [z^{-n} Z_n(z)] = (-1)^m z^{-n-m} Z_{n+m}(z)$$

Here $Z_n(z)$ is a fixed arbitrary linear combination of $J_n(x)$, $N_n(x)$, $H_n^{(1)}(x)$ or $H_n^{(2)}(x)$. Specifically, we have

$$J_1(z) = -J'_0(z)$$

which is frequently used in the analysis in this textbook.

Some useful indefinite integrals with solutions to the Bessel differential equation, which are often used in the text, are ($n = 0, 1, 2, \dots$)

$$\int x^{n+1} Z_n(x) \, dx = x^{n+1} Z_{n+1}(x) = -x^{n+1} \left(Z'_n(x) - \frac{n}{x} Z_n(x) \right)$$

$$\int x^{-n+1} Z_n(x) \, dx = -x^{-n+1} Z_{n-1}(x) = -x^{-n+1} \left(Z'_n(x) + \frac{n}{x} Z_n(x) \right)$$

$$\int x (Z_n(x))^2 \, dx = \frac{x^2}{2} [(Z_n(x))^2 - Z_{n-1}(x) Z_{n+1}(x)]$$

$$= \frac{x^2}{2} (Z'_n(x))^2 + \frac{1}{2} (x^2 - n^2) (Z_n(x))^2$$

⁴These recursion relations hold for non-integer values of n , t.ex. $n = 1/2$. The index m , however, must be an integer.

As above, $Z_n(x)$ is an arbitrary linear combination of $J_n(x)$, $N_n(x)$, $H_n^{(1)}(x)$ or $H_n^{(2)}(x)$. Some additional — more complex — but useful determined integrals are ($n = 0, 1, 2, \dots, m = 0, 1, 2, \dots$)

$$\begin{aligned} \int \left[(\alpha^2 - \beta^2)x - \frac{m^2 - n^2}{x} \right] Z_m(\alpha x) Y_n(\beta x) dx &= \beta x Z_m(\alpha x) Y_{n-1}(\beta x) \\ &\quad - \alpha x Z_{m-1}(\alpha x) Y_n(\beta x) + (m - n) Z_m(\alpha x) Y_m(\beta x) \\ \int x Z_m(\alpha x) Y_m(\beta x) dx &= \frac{\beta x Z_m(\alpha x) Y_{m-1}(\beta x) - \alpha x Z_{m-1}(\alpha x) Y_m(\beta x)}{\alpha^2 - \beta^2} \\ \int \frac{Z_m(\alpha x) Y_n(\alpha x)}{x} dx &= \alpha x \frac{Z_{m-1}(\alpha x) Y_n(\alpha x) - Z_m(\alpha x) Y_{n-1}(\alpha x)}{m^2 - n^2} - \frac{Z_m(\alpha x) Y_n(\alpha x)}{m + n} \end{aligned}$$

Here, $Z_n(\alpha x)$ and $Y_n(\beta x)$ is an arbitrary linear combination of $J_n(x)$, $N_n(x)$, $H_n^{(1)}(x)$ or $H_n^{(2)}(x)$.

For Bessel functions, $J_n(z)$, Neumann functions, $N_n(z)$, and Hankel functions, $H_n^{(1)}(z)$ or $H_n^{(2)}(z)$, we have for a complex argument z

$$\begin{cases} J_n(z^*) = (J_n(z))^* & H_n^{(1)}(z^*) = (H_n^{(2)}(z))^* \\ N_n(z^*) = (N_n(z))^* & H_n^{(2)}(z^*) = (H_n^{(1)}(z))^* \end{cases}$$

The Wronskian relation for $J_n(z)$ and $N_n(z)$ is

$$J_n(z)N_n'(z) - J_n'(z)N_n(z) = \frac{2}{\pi z}$$

and for $J_n(z)$ and $H_n^{(1)}(z)$ it is

$$J_n(z)H_n^{(1)'}(z) - J_n'(z)H_n^{(1)}(z) = \frac{2i}{\pi z} \quad (\text{B.6})$$

The Graf addition theorem for Bessel functions is useful. Let $Z_n(x)$ be any linear combination of $J_n(x)$, $N_n(x)$, $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$. The Graf addition theorem is

$$Z_n(w) \begin{pmatrix} \cos n\phi \\ \sin n\phi \end{pmatrix} = \sum_{k=-\infty}^{\infty} Z_{n+k}(u) J_k(v) \begin{pmatrix} \cos k\alpha \\ \sin k\alpha \end{pmatrix}, \quad |ve^{\pm i\alpha}| < |u|$$

where w is

$$w = \sqrt{u^2 + v^2 - 2uv \cos \alpha}$$

The relations between the different quantities w , u , v , α , and ϕ are shown in Figure B.1.

B.2 Spherical Bessel and Hankel functions

The spherical Bessel and Hankel functions show up in scattering problems, especially when we express a solution in the spherical coordinate system (r, θ, ϕ) .

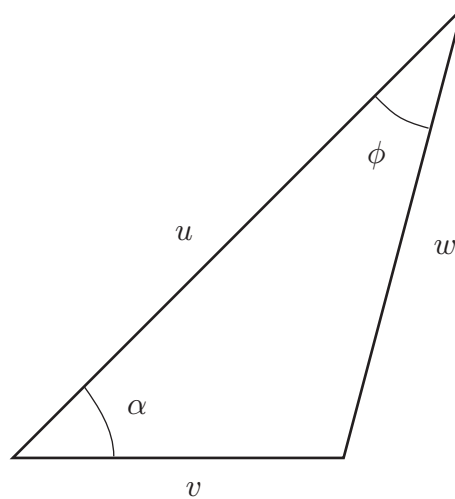


Figure B.1: The geometry of the quantities in the addition theorem.

The spherical Bessel differential equation is

$$z^2 \frac{d^2}{dz^2} Z_n(z) + 2z \frac{d}{dz} Z_n(z) + (z^2 - n(n+1)) Z_n(z) = 0 \quad (\text{B.7})$$

where n is assumed to be a non-negative integer and the argument z a complex number.

A solution to this differential equation is the spherical Bessel functions $j_n(z)$, which are defined by the power series expansion⁵

$$j_n(z) = 2^n z^n \sum_{k=0}^{\infty} \frac{(-1)^k (k+n)!}{k! (2k+2n+1)!} z^{2k} \quad (\text{B.8})$$

This solution is real-valued for real arguments, it is regular at the origin, $z = 0$, and its power series is absolutely convergent in the entire complex plane. From the power series, we also conclude that $j_n(z)$ is an even function for even integers n and an odd function for odd n , *i.e.*,

$$j_n(-z) = (-1)^n j_n(z)$$

The other, linearly independent solution, which is real-valued for real arguments, is the spherical Neumann function $y_n(z)$. Its power series is

$$y_n(z) = \frac{(-1)^{n+1} 2^n \pi^{1/2}}{z^{n+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k-n-1/2)!} \left(\frac{z}{2}\right)^{2k} = \frac{(-1)^{n+1}}{2^n z^{n+1}} \sum_{k=0}^{\infty} \frac{(-1)^k (k-n)!}{k! (2k-2n)!} z^{2k}$$

⁵Between the spherical Bessel function, $j_l(z)$, and the cylindrical Bessel function, $J_n(z)$, see Appendix B.1, there is a relation

$$j_l(z) = \left(\frac{\pi}{2z}\right)^{1/2} J_{l+1/2}(z)$$

Also between the other type of solutions to the spherical Bessel equation and the corresponding solutions to the cylindrical Bessel equation, (B.1), there are similar relations.

We see that this solution is singular at the origin $z = 0$.

In scattering problems, we often use linear combinations of the spherical Bessel and Neumann functions. These are the spherical Hankel functions, $h_n^{(1)}(z)$ and $h_n^{(2)}(z)$ of the first and second kind, respectively. These are defined as

$$\begin{aligned} h_n^{(1)}(z) &= j_n(z) + iy_n(z) \\ h_n^{(2)}(z) &= j_n(z) - iy_n(z) \end{aligned}$$

The spherical Bessel, Neumann, and Hankel functions for $l = 0$ are

$$\begin{cases} j_0(z) = \frac{\sin z}{z} \\ y_0(z) = -\frac{\cos z}{z} \end{cases} \quad \begin{cases} h_0^{(1)}(z) = -\frac{i}{z}e^{iz} \\ h_0^{(2)}(z) = \frac{i}{z}e^{-iz} \end{cases}$$

and for $l = 1$ they are

$$\begin{cases} j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z} \\ y_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z} \end{cases} \quad \begin{cases} h_1^{(1)}(z) = e^{iz} \left(-\frac{1}{z} - \frac{i}{z^2} \right) \\ h_1^{(2)}(z) = e^{-iz} \left(-\frac{1}{z} + \frac{i}{z^2} \right) \end{cases}$$

From the power series representation in (B.8) we obtain for small arguments

$$\begin{cases} j_n(z) = \frac{2^n n! z^n}{(2n+1)!} + O(z^{n+2}) \\ h_n^{(1)}(z) = -i \frac{(2n)!}{2^n n! z^{n+1}} + O(z^{-n+1}) \\ (zj_n(z))' = \frac{2^n (n+1)! z^n}{(2n+1)!} + O(z^{n+2}) \\ (zh_n^{(1)}(z))' = i \frac{(2n)!}{2^n (n-1)! z^{n+1}} + O(z^{-n+1}) \end{cases} \quad \text{as } z \rightarrow 0 \quad (\text{B.9})$$

Specifically, for $l = 1$

$$\begin{cases} j_1(z) \approx \frac{z}{3}, \text{ as } z \rightarrow 0 \\ h_1^{(1)}(z) \approx \frac{1}{iz^2}, \text{ as } z \rightarrow 0 \end{cases} \quad \begin{cases} \frac{(zj_1(z))'}{z} \approx \frac{2}{3}, \text{ as } z \rightarrow 0 \\ \frac{(zh_1^{(1)}(z))'}{z} \approx -\frac{1}{iz^3}, \text{ as } z \rightarrow 0 \end{cases}$$

An alternative expansion of the spherical Bessel and Neumann functions in finite trigonometric series are:

$$\begin{cases} j_n(z) = \frac{1}{z} \left\{ p_n(z) \sin \left(z - \frac{n\pi}{2} \right) + q_n(z) \cos \left(z - \frac{n\pi}{2} \right) \right\} \\ y_n(z) = \frac{1}{z} \left\{ q_n(z) \sin \left(z - \frac{n\pi}{2} \right) - p_n(z) \cos \left(z - \frac{n\pi}{2} \right) \right\} \end{cases} \quad (\text{B.10})$$

The explicit for of the finite sums are

$$\begin{cases} p_n(z) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(n+2k)!}{(2k)!(n-2k)!} \frac{1}{(2z)^{2k}} = 1 - \frac{b_n}{z^2} + O(z^{-4}) \\ q_n(z) = \sum_{k=0}^{[(n-1)/2]} (-1)^k \frac{(n+2k+1)!}{(2k+1)!(n-2k-1)!} \frac{1}{(2z)^{2k+1}} = \frac{a_n}{z} + O(z^{-3}) \end{cases} \quad (\text{B.11})$$

where $[\cdot]$ denotes the integer part of the argument, and where $a_n = (n+1)n/2$ and $b_n = (n+2)(n+1)n(n-1)/8$. Note that these series are finite and that the series, by definition, is zero if the upper limit of summation is negative.

Similarly, for the spherical Hankel functions we have

$$\begin{cases} h_n^{(1)}(z) = \frac{e^{iz-i(n+1)\pi/2}}{z} \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!} \frac{1}{(-2iz)^k} = \frac{e^{iz-i(n+1)\pi/2}}{z} (p_n(z) + iq_n(z)) \\ h_n^{(2)}(z) = \frac{e^{-iz+i(n+1)\pi/2}}{z} \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!} \frac{1}{(2iz)^k} = \frac{e^{-iz+i(n+1)\pi/2}}{z} (p_n(z) - iq_n(z)) \end{cases} \quad (\text{B.12})$$

In scattering problems another combination of spherical Bessel and Hankel functions occur, *i.e.*,

$$\begin{cases} \frac{(zj_n(z))'}{z} = \frac{1}{z} \left\{ P_n(z) \sin\left(z - \frac{n\pi}{2}\right) + Q_n(z) \cos\left(z - \frac{n\pi}{2}\right) \right\} \\ \frac{(zh_n^{(1)}(z))'}{z} = \frac{e^{iz-i(n+1)\pi/2}}{z} (P_n(z) + iQ_n(z)) \\ \frac{(zh_n^{(2)}(z))'}{z} = \frac{e^{-iz+i(n+1)\pi/2}}{z} (P_n(z) - iQ_n(z)) \end{cases} \quad (\text{B.13})$$

where

$$\begin{cases} P_n(z) = p'_n(z) - q_n(z) = -\frac{a_n}{z} + O(z^{-3}) \\ Q_n(z) = q'_n(z) + p_n(z) = 1 - \frac{a_n + b_n}{z^2} + O(z^{-4}) \end{cases} \quad (\text{B.14})$$

Between different orders of spherical Bessel functions there are recursion relations. Some of the important are ($n = 0, 1, 2, \dots, m = 0, 1, 2, \dots$)

$$\begin{cases} f_{n-1}(z) + f_{n+1}(z) = \frac{2n+1}{z} f_n(z) \\ nf_{n-1}(z) - (n+1)f_{n+1}(z) = (2n+1)f'_n(z) \\ nf_n(z) - zf_{n+1}(z) = zf'_n(z) \\ \left(\frac{d}{zdz}\right)^m [z^{n+1}f_n(z)] = z^{n-m+1}f_{n-m}(z) \\ \left(\frac{d}{zdz}\right)^m [z^{-n}f_n(z)] = (-1)^m z^{-n-m}f_{n+m}(z) \end{cases} \quad (\text{B.15})$$

Here $f_n(z)$ is a spherical Bessel function, $j_n(z)$, a spherical Neumann function, $y_n(z)$, or the Hankel functions $h_n^{(1)}(z)$ or $h_n^{(2)}(z)$.

Useful, especially in the analysis of scattering by spherical vector waves, is also:

$$\frac{(zf_n(z))'}{z} = \frac{f_n(z)}{z} + f_n'(z) = \frac{(n+1)f_{n-1}(z) - nf_{n+1}(z)}{2n+1} = \frac{(n+1)f_n(z)}{z} - f_{n+1}(z) \quad (\text{B.16})$$

The Wronskian relation for $j_n(z)$ and $y_n(z)$ is

$$j_n(z)y_n'(z) - j_n'(z)y_n(z) = \frac{1}{z^2} \quad (\text{B.17})$$

and for $j_n(z)$ and $h_n^{(1)}(z)$ it is

$$j_n(z)h_n^{(1)'}(z) - j_n'(z)h_n^{(1)}(z) = \frac{i}{z^2} \quad (\text{B.18})$$

Lemma B.1. *The functions $h_n^{(1)}(z)$ and $h_n^{(1)'}(z)$ have no real zeros.*

Proof: By inspection, we conclude that $x = 0$ is not a root to neither $h_n^{(1)}(z)$ nor $h_n^{(1)'}(z)$. A root $x \neq 0$ to $h_n^{(1)}(x) = 0$ implies that both the real and imaginary parts must be zero, *i.e.*, $j_n(x) = y_n(x) = 0$, which is in conflict with (B.17). Similarly, a root $x \neq 0$ to $h_n^{(1)'}(x) = 0$ implies that $j_n'(x) = y_n'(x) = 0$, which again is in conflict with (B.17). \square

The addition theorem for spherical Bessel functions appears in *e.g.*, the some expansions of the Green's function in free space. Let $f_n(x)$ be any linear combination of $j_n(x)$, $y_n(x)$, $h_n^{(1)}(x)$ and $h_n^{(2)}(x)$, then the Gegenbauer's addition theorem is

$$\frac{f_n(w)}{w^n} = (2n-1)!! \sum_{k=0}^{\infty} (2k+2n+1) \frac{f_{n+k}(u)}{u^n} \frac{j_{n+k}(v)}{v^n} C_k^{(n+1/2)}(\cos \alpha)$$

where $C_k^{(n+1/2)}(\cos \alpha)$ are the Gegenbauer polynomials, which also can written as

$$C_k^{(n+1/2)}(x) = \frac{1}{(2n-1)!!} (1-x^2)^{-n/2} P_{n+k}^k(x)$$

We have

$$\frac{f_n(w)}{w^n} = \sum_{k=0}^{\infty} (2k+2n+1) \frac{f_{n+k}(u)}{u^n} \frac{j_{n+k}(v)}{v^n} \frac{P_{n+k}^n(\cos \alpha)}{\sin^n \alpha} \quad (\text{B.19})$$

The relations between the different quantities w , u , v , and α are depicted in Figure B.1.

B.2.1 Integral representations

The spherical Bessel and Hankel functions have the following useful integral representations:

$$j_n(z) = \frac{1}{2i^n} \int_{-1}^1 P_n(t) e^{itz} dt \quad (\text{B.20})$$

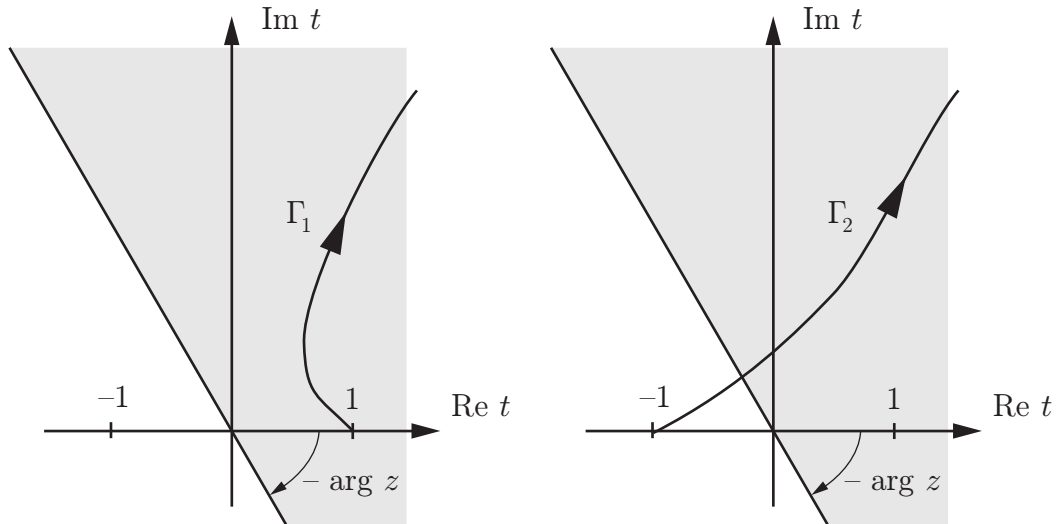


Figure B.2: The complex integration contours Γ_1 and Γ_2 . The contours Γ_1 and Γ_2 can end at infinity anywhere in the shaded half plane.

and

$$\begin{cases} h_n^{(1)}(z) = -\frac{1}{i^n} \int_{\Gamma_1} P_n(t) e^{itz} dt \\ h_n^{(2)}(z) = \frac{1}{i^n} \int_{\Gamma_2} P_n(t) e^{itz} dt \end{cases} \quad (\text{B.21})$$

where $P_n(t)$ are the Legendre polynomials, and the complex contours Γ_1 and Γ_2 are defined in Figure B.2, which for a real argument z can be specialized as in Figure B.3.

We prove these identities by the use of the differential equations for the spherical Bessel functions and the Legendre polynomials, *i.e.*,

$$z^2 f_n''(z) + 2z f_n'(z) + (z^2 - n(n+1)) f_n(z) = 0$$

and

$$(1-t^2)P_n''(t) - 2tP_n'(t) + n(n+1)P_n(t) = 0 \quad (\text{B.22})$$

respectively.

First prove that an integral

$$f_n(z) = C_n \int_{\Gamma} P_n(t) e^{itz} dt \quad (\text{B.23})$$

satisfies the differential equation for the spherical Bessel functions. The contour Γ and the constant C_n are determined below. We get

$$\begin{aligned} & \left[z^2 \frac{d^2}{dz^2} + 2z \frac{d}{dz} + z^2 - n(n+1) \right] \int_{\Gamma} P_n(t) e^{itz} dt \\ &= \int_{\Gamma} [-t^2 z^2 + 2izt + z^2 - n(n+1)] P_n(t) e^{itz} dt \end{aligned}$$

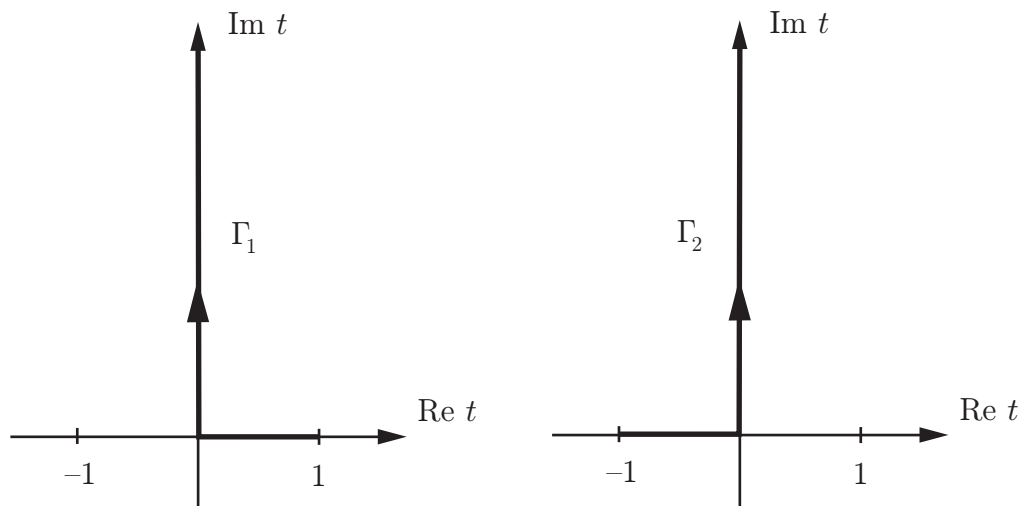


Figure B.3: The complex integration contours Γ_1 and Γ_2 for a real argument z .

The use of (B.22) and integration by parts give

$$\begin{aligned}
 & \left[z^2 \frac{d^2}{dz^2} + 2z \frac{d}{dz} + z^2 - n(n+1) \right] \int_{\Gamma} P_n(t) e^{itz} dt \\
 &= \int_{\Gamma} [-t^2 z^2 + 2izt + z^2] P_n(t) e^{itz} dt + \int_{\Gamma} [(1-t^2)P_n''(t) - 2tP_n'(t)] e^{itz} dt \\
 &= \int_{\Gamma} [-t^2 z^2 + 2izt + z^2 + 2itz + 2] P_n(t) e^{itz} dt \\
 &\quad + \{(1-t^2)P_n'(t) - 2tP_n(t)\} e^{itz} \Big|_{\Gamma} - \int_{\Gamma} [iz(1-t^2) - 2t] P_n'(t) e^{itz} dt \\
 &= (1-t^2) \{P_n'(t) - izP_n(t)\} e^{itz} \Big|_{\Gamma}
 \end{aligned}$$

and we see that the integral in (B.23) is a solution to the differential equation for the spherical Bessel functions provided

$$(1-t^2) \{P_n'(t) - izP_n(t)\} e^{itz} \Big|_{\Gamma} = 0$$

We now choose the contour Γ such that the contributions at the endpoints vanish. This can be done in many ways. The first, obvious way is to let the contour Γ be the linear segment $[-1, 1]$. Then, at the endpoints, $z = \pm 1$, the contributions vanish. The solution in (B.23) then is regular at $z = 0$, it must be a multiple of $j_n(z)$, *i.e.*,

$$j_n(z) = C_n \int_{-1}^1 P_n(t) e^{itz} dt$$

To determine the constant C_n , we expand the exponential in the integrand and identify lowest non-vanishing power of z on both sides in the identity. The spherical Bessel function has an expansion

$$j_n(z) = \frac{2^n n!}{(2n+1)!} z^n + O(z^{n+2})$$

The integral has the expansion

$$\int_{-1}^1 P_n(t) e^{itz} dt = \sum_{k=n}^{\infty} \frac{i^k z^k}{k!} \int_{-1}^1 P_n(t) t^k dt$$

where we have used the orthogonality of the Legendre polynomials

$$\int_{-1}^1 P_n(t) t^k dt = 0, \quad k = 0, 1, 2, \dots, n-1$$

to show that the sum starts at $k = n$. The use of the series expansion of the Legendre polynomials in Appendix C.1 gives

$$t^n = \frac{2^n n! n!}{(2n)!} P_n(t) + \sum_{k=0}^{n-1} b_k(n) P_k(t)$$

and therefore, by orthogonality, the lowest non-vanishing power in z of the integral is

$$\int_{-1}^1 P_n(t) e^{itz} dt = \frac{i^n z^n}{n!} \frac{2^n n! n!}{(2n)!} \frac{2}{2n+1} + O(z^{n+2}) = \frac{i^n 2^{n+1} n!}{(2n+1)!} z^n + O(z^{n+2})$$

From this we can conclude that the undetermined coefficient C_n is

$$C_n 2i^n = 1$$

and (B.20) is proved.

Another choice of the contour Γ leads to the spherical Hankel functions. Let Γ_1 and Γ_2 start at $t = \pm 1$, respectively, and end at infinity $t = \infty e^{i\phi}$, where $\phi \in (-\arg z, \pi - \arg z)$. Then again, the contributions from the endpoints vanish, since $\text{Im}(tz) > 0$ at the endpoint at infinity. To determine what type of solution it is, we determine the behavior of these solutions as $|z| \rightarrow \infty$. The asymptotic behavior at large arguments is obtained by integration by parts. We get

$$\begin{cases} \int_{\Gamma_1} P_n(t) e^{itz} dt = \frac{1}{iz} P_n(t) e^{itz} \Big|_{\Gamma_1} (1 + O(z^{-1})) = -\frac{1}{iz} e^{iz} (1 + O(z^{-1})) \\ \int_{\Gamma_2} P_n(t) e^{itz} dt = \frac{1}{iz} P_n(t) e^{itz} \Big|_{\Gamma_2} (1 + O(z^{-1})) = -\frac{1}{iz} (-1)^n e^{-iz} (1 + O(z^{-1})) \end{cases}$$

A comparison with the leading behavior for the spherical Hankel functions in (B.12)

$$\begin{cases} h_n^{(1)}(z) = \frac{e^{iz-i(n+1)\pi/2}}{z} (1 + O(z^{-1})) \\ h_n^{(2)}(z) = \frac{e^{-iz+i(n+1)\pi/2}}{z} (1 + O(z^{-1})) \end{cases}$$

determines the common constant, and completes the derivation of the integral representations in (B.21).

B.2.2 Related functions

It is also convenient to identify the differential equation for a spherical Bessel function times a power of z , *i.e.*, $F_{\nu n}(z) = z^{\nu} f_n(z)$, where $\nu = 0, \pm 1, \pm 2, \dots$ is an integer. The function $F_{\nu n}(z)$ satisfies

$$z^2 \frac{d^2}{dz^2} F_{\nu n}(z) + 2z(1-\nu) \frac{d}{dz} F_{\nu n}(z) + (z^2 - n(n+1) + \nu(\nu+1) - 2\nu) F_{\nu n}(z) = 0 \quad (\text{B.24})$$

The solution for $\nu = 1$ becomes particularly simple

$$z^2 \frac{d^2}{dz^2} F_{1n}(z) + (z^2 - n(n+1)) F_{1n}(z) = 0 \quad (\text{B.25})$$

Related to the spherical Bessel and Hankel functions are the Riccati-Bessel functions. These are usually defined as [17]

$$\begin{cases} \psi_n(z) = z j_n(z) \\ \xi_n(z) = z h_n^{(1)}(z) \end{cases} \quad (\text{B.26})$$

These functions satisfy the Riccati-Bessel differential equation is

$$z^2 \frac{d^2}{dz^2} Z_n(z) + (z^2 - n(n+1)) Z_n(z) = 0$$

where n is assumed to be a non-negative integer and the argument z a complex number.

The Wronskian relation for $\psi_n(z)$ and $\xi_n(z)$ is

$$\psi_n(z) \xi_n'(z) - \psi_n'(z) \xi_n(z) = i$$

Lemma B.2. *The functions $\xi_n(z)$, and $\xi_n'(z)$ have no real zeros.*

Proof: By inspection, we conclude that $x = 0$ is not a root to neither $\xi_n(z)$, nor $\xi_n'(z)$. The lack of a root $x \neq 0$ to $\xi_n(z) = z h_n^{(1)}(z)$ follows directly from Lemma B.1 on page 77. Finally, the function $\xi_n'(x)$ has no real zeros, since a real root, $x \neq 0$, to $\xi_n(x)$ implies that $j_n(x) + x j_n'(x) = y_n(x) + x y_n'(x) = 0$, which is in conflict with (B.17). \square

Many relations are derived from the properties of the spherical Bessel and Hankel functions above. For example, a useful recursion relation of the Riccati-Bessel functions is, see (B.15)

$$\begin{cases} f_{n+1}(z) = (2n+1) \frac{f_n(z)}{z} - f_{n-1}(z) \\ f_n'(z) = (n+1) \frac{f_n(z)}{z} - f_{n+1}(z) = f_{n-1}(z) - n \frac{f_n(z)}{z} \end{cases} \quad n = 0, 1, 2, \dots$$

where $f_n(z)$ is any combination of the functions $\psi_n(z)$ or $\xi_n(z)$.

Orthogonal polynomials

In this appendix, we define the orthogonal polynomials that are used in the book, and, moreover, we list some of their most important properties. The derivations and more details are found in *e.g.*, , Ref. 1.

C.1 Legendre polynomials

We define the Legendre polynomials, $P_n(x)$, from the generating function $F(x, t) = (1 - 2xt + t^2)^{-1/2}$, *i.e.*,

$$\frac{1}{(1 - 2xt + t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \quad |t| < 1$$

An alternative way is by the Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \tag{C.1}$$

This leads to a power series expansion

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n - 2k)!}{2^n k!(n - k)!(n - 2k)!} x^{n-2k}$$

The first polynomials are:

$$\begin{cases} P_0(x) = 1 \\ P_1(x) = x \\ P_2(x) = (3x^2 - 1)/2 \\ P_3(x) = (5x^3 - 3x)/2 \end{cases}$$

Between different orders of the Legendre polynomials there are relations, *viz.* the

recursion relations. The most important are:

$$\begin{cases} (n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x) \\ nP_n(x) = xP'_n(x) - P'_{n-1}(x) \\ (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \\ (2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \\ (1-x^2)P'_n(x) = (n+1)(xP_n(x) - P_{n+1}(x)) = n(P_{n-1}(x) - xP_n(x)) \end{cases}$$

With these recursion relations, we easily prove

$$P_n(1) = 1$$

for all $n = 0, 1, 2, \dots$, and

$$P'_n(1) = \frac{n(n+1)}{2}$$

The value at the origin is

$$P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} = (-1)^n \frac{(2n-1)!!}{(2n)!!}, \quad P_{2n+1}(0) = 0, \quad n = 0, 1, 2, 3, \dots$$

Moreover, we have

$$P_n(-x) = (-1)^n P_n(x)$$

The Legendre polynomials are orthogonal on the interval $[-1, 1]$, *i.e.*,

$$\int_{-1}^1 P_n(x)P_{n'}(x) \, dx = \frac{2\delta_{nn'}}{2n+1}$$

and they satisfy the differential equation

$$(1-x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0$$

Moreover, the set $\{P_n(x)\}_{n=0}^{\infty}$ is a complete orthogonal system on the interval $[-1, 1]$. This means that every square integrable function on the interval $[-1, 1]$ has a convergent Fourier series. The convergence is in general only in norm.

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x), \quad x \in [-1, 1]$$

where the (Fourier) coefficients a_n are determined by the integrals

$$a_n = \frac{2n+1}{2} \int_{-1}^1 f(x)P_n(x) \, dx$$

Another useful integral that involves Legendre polynomials is

$$\int_{-1}^1 xP_n(x)P_{n'}(x) \, dx = \frac{2(n+1)}{(2n+1)(2n+3)}\delta_{n',n+1} + \frac{2n}{(2n-1)(2n+1)}\delta_{n',n-1}$$

Appendix D

Spherical harmonics

In scattering theory, the spherical harmonics play an important role, and in this appendix, we define these functions and also present some of their more important properties. Specifically, the transformation properties between spherical wave and plane waves are derived.

D.1 Associated Legendre functions

The associated Legendre functions, defined for positive integers $l, m \geq 0$, are denoted $P_l^m(x)$ and defined by [1]

$$P_l^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad x \in [-1, 1]$$

where $P_l(x)$ is the Legendre polynomial defined in Appendix C. For a negative integer value of m , we use $(m = -1, -2, \dots)$

$$P_l^m(x) = (-1)^{|m|} \frac{(l - |m|)!}{(l + |m|)!} P_l^{|m|}(x), \quad x \in [-1, 1] \quad (\text{D.1})$$

The integer l takes non-negative integer values, *i.e.*, $l = 0, 1, 2, 3, \dots$, and the integer m takes values in the interval $m = -l, -l + 1, \dots, -1, 0, 1, \dots, l$, since $|m| > l$ leads to $P_l^m(x) = 0$. We also note that

$$P_l^0(x) = P_l(x)$$

and

$$P_l^m(1) = \delta_{m0}$$

Other useful limits are

$$\lim_{x \rightarrow 1} \sqrt{1 - x^2} P_l^{m'}(x) = -\delta_{m1} \frac{l(l+1)}{2}$$

and for $(m > 0)$

$$\lim_{x \rightarrow 1} \frac{P_l^m(x)}{\sqrt{1 - x^2}} = \delta_{m1} \frac{l(l+1)}{2}$$

The value at the origin is

$$P_l^m(0) = \begin{cases} (-1)^{(l-m)/2} \frac{(l+m)!}{2^l \left(\frac{l-m}{2}\right)! \left(\frac{l+m}{2}\right)!}, & l+m \text{ even} \\ 0, & l+m \text{ odd} \end{cases}$$

In many applications it is convenient to introduce an angle $\theta \in [0, \pi]$ instead of the variable x . Define $x = \cos \theta$, and the definition of the associated Legendre functions become ($m \geq 0$)

$$P_l^m(\cos \theta) = \sin^m \theta \frac{d^m}{d(\cos \theta)^m} P_l(\cos \theta), \quad \theta \in [0, \pi]$$

The functions are either even or odd, depending on the value of $l + m$.

$$P_l^m(-x) = (-1)^{l+m} P_l^m(x)$$

As the Legendre polynomials, the associated Legendre functions are orthogonal over the interval $[-1, 1]$.

$$\int_{-1}^1 P_l^m(x) P_{l'}^m(x) dx = \int_0^\pi P_l^m(\cos \theta) P_{l'}^m(\cos \theta) \sin \theta d\theta = \frac{2\delta_{ll'}}{2l+1} \frac{(l+m)!}{(l-m)!}$$

and

$$\begin{aligned} \int_0^\pi \left(\frac{\partial P_l^m(\cos \theta)}{\partial \theta} \frac{\partial P_{l'}^m(\cos \theta)}{\partial \theta} + m^2 \frac{P_l^m(\cos \theta)}{\sin \theta} \frac{P_{l'}^m(\cos \theta)}{\sin \theta} \right) \sin \theta d\theta \\ = \frac{2l(l+1)\delta_{ll'}}{2l+1} \frac{(l+m)!}{(l-m)!} \end{aligned}$$

and

$$\int_0^\pi \left(\frac{\partial P_l^m(\cos \theta)}{\partial \theta} \frac{P_{l'}^m(\cos \theta)}{\sin \theta} + \frac{P_l^m(\cos \theta)}{\sin \theta} \frac{\partial P_{l'}^m(\cos \theta)}{\partial \theta} \right) \sin \theta d\theta = 0$$

Note that the m index is the same in both functions in the integrand. There is also another orthogonality relation for the associated Legendre functions, but this holds for an l index that is the same, $m \neq 0$, and with a weight function $(1-x^2)^{-1}$.

$$\int_{-1}^1 \frac{P_l^m(x) P_{l'}^m(x)}{1-x^2} dx = \frac{\delta_{ll'}}{m} \frac{(l+m)!}{(l-m)!}$$

Between different associated Legendre function there exist relations. These are either recursion relations in the index l or the azimuthal index m or a combination of these indices. Some of the useful recursion relations are:

$$\begin{cases} P_l^{m+1}(x) - \frac{2mx}{(1-x^2)^{1/2}} P_l^m(x) + (l+m)(l-m+1) P_l^{m-1}(x) = 0 \\ (2l+1)(1-x^2)^{1/2} P_l^m(x) = P_{l+1}^{m+1}(x) - P_{l-1}^{m+1}(x) \\ \quad = (l+m)(l+m-1) P_{l-1}^{m-1}(x) - (l-m+1)(l-m+2) P_{l+1}^{m-1}(x) \\ (1-x^2)^{1/2} P_l^{m'}(x) = \frac{1}{2} P_l^{m+1}(x) - \frac{1}{2} (l+m)(l-m+1) P_l^{m-1}(x) \end{cases}$$

Useful is also

$$\begin{cases} (1-x^2)P_l^{m'}(x) = (l+m)P_{l-1}^m(x) - lxP_l^m(x) \\ \qquad \qquad \qquad = (l+1)xP_l^m(x) - (l-m+1)P_{l+1}^m(x) \\ (l-m+1)P_{l+1}^m(x) = (2l+1)xP_l^m(x) - (l+m)P_{l-1}^m(x) \end{cases}$$

The value of the derivative at the origin is

$$P_l^{m'}(0) = \begin{cases} 0, & l+m \text{ even} \\ (l+m)P_{l-1}^m(0), & l+m \text{ odd} \end{cases}$$

and

$$P_l^{m''}(0) = \begin{cases} (m^2 - l(l+1))P_l^m(0), & l+m \text{ even} \\ 0, & l+m \text{ odd} \end{cases}$$

D.2 Spherical harmonics

The spherical harmonics are denoted $Y_{lm}(\theta, \phi)$ and they are defined by [1]

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi} = C_{lm} P_l^m(\cos\theta) e^{im\phi}$$

where the indices l, m take the following values:

$$l = 0, 1, 2, \dots, \quad m = -l, -l+1, \dots, -1, 0, 1, \dots, l-1, l$$

The normalization factor C_{lm} used in this textbook is

$$C_{lm} = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}$$

The extra factor $(-1)^m$ is called the Condon-Shortley phase. This extra factor and the expression of the associated Legendre functions for negative m -values in (D.1) gives us an alternative expression of the spherical harmonics, *i.e.*,

$$Y_{lm}(\theta, \phi) = \left(-\frac{m}{|m|}\right)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos\theta) e^{im\phi}$$

where

$$\left(-\frac{m}{|m|}\right)^m = \begin{cases} (-1)^m, & m > 0 \\ 1, & m = 0 \\ 1, & m < 0 \end{cases}$$

This definition employs only associated Legendre functions with non-negative m -values.

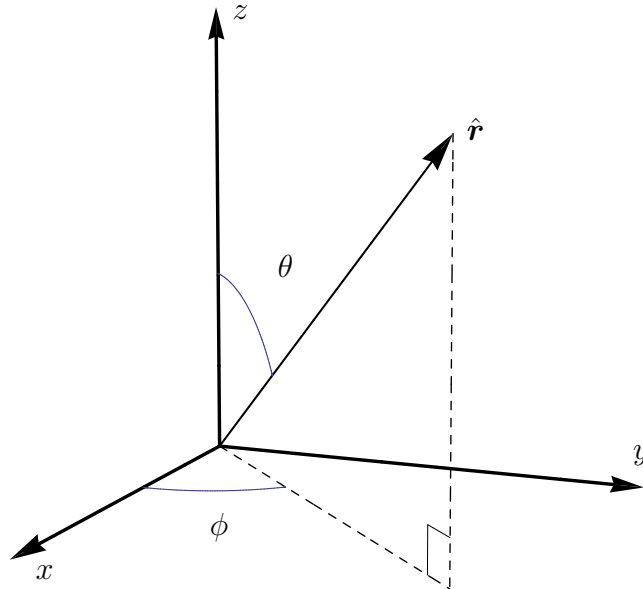


Figure D.1: Definition of the spherical angles θ and ϕ .

Real vs complex notation

In the literature, there are several ways of defining the spherical harmonics, each with its pros and cons. Commonly used are the complex combinations [1]

$$Y_{lm}(\theta, \phi) = C_{lm} P_l^m(\cos \theta) e^{im\phi}$$

The real combinations of the azimuth functions are [12]

$$Y_{\sigma ml}(\theta, \phi) = C'_{lm} P_l^m(\cos \theta) \begin{cases} \cos m\phi \\ \sin m\phi \end{cases}$$

where $m = 0, 1, \dots, l-1, l$ and $\sigma = e, o$ (even or odd in ϕ). The transformations between the two are ($m \geq 0$)

$$\begin{cases} Y_{l\pm m} = \frac{C_{l\pm m}}{C'_{lm}} (Y_{e ml} \pm i Y_{o ml}) \\ Y_{\sigma ml} = \frac{C'_{lm}}{2C_{lm}} \begin{cases} Y_{lm} + \frac{C_{lm}}{C_{l-m}} Y_{l-m} \\ -i Y_{lm} + i \frac{C_{lm}}{C_{l-m}} Y_{l-m} \end{cases} \end{cases}$$

of the spherical harmonics

$$Y_{lm}(-\hat{\mathbf{r}}) = (-1)^l Y_{lm}(\hat{\mathbf{r}})$$

The definitions of the spherical harmonics differ between textbooks, and it is recommended that the explicit definition of these functions is checked carefully before use. It is mostly the use of different phase factors that causes problems. In this text we adopt the Condon-Shortley phase factor, since it is the most widely used phase factor in the literature. The two main alternatives of definitions of the spherical harmonics are given in the text box. The normalization coefficient often varies between different textbooks.

Instead of writing the angles θ and ϕ in the argument, we often use the unit vector $\hat{\mathbf{r}} = \hat{\mathbf{x}} \sin \theta \cos \phi + \hat{\mathbf{y}} \sin \theta \sin \phi + \hat{\mathbf{z}} \cos \theta$, see Figure D.1, and the spherical harmonics are often denoted $Y_{lm}(\hat{\mathbf{r}})$.

Reflection of the unit position vector in the origin $\hat{\mathbf{r}} \rightarrow -\hat{\mathbf{r}}$, *i.e.*, $\theta \rightarrow \pi - \theta$, $\phi \rightarrow \phi + \pi$, implies the parity conditions

Special values, which are important in scattering problems, are

$$Y_{lm}(\hat{\mathbf{z}}) = \delta_{m0} \sqrt{\frac{2l+1}{4\pi}} \quad (\text{D.2})$$

Moreover, the lowest order spherical harmonics are

$$Y_{00}(\hat{\mathbf{r}}) = \sqrt{\frac{1}{4\pi}},$$

$$\begin{cases} Y_{10}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \cos \theta \\ Y_{1\pm 1}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} \end{cases}$$

Frequently, we need to change the sign in the exponential, which brings about the notation

$$Y_{lm}^\dagger(\theta, \phi) = C_{lm} P_l^m(\cos \theta) e^{-im\phi}$$

Note that the dagger, (\dagger), is identical to the complex conjugate, denoted ($*$), of the spherical harmonics, provided the angles θ and ϕ are real numbers,¹ *i.e.*,

$$Y_{lm}^\dagger(\theta, \phi) = Y_{lm}^*(\theta, \phi), \quad \theta, \phi \in \mathbb{R}$$

We also have

$$Y_{l-m}(\theta, \phi) = (-1)^m Y_{lm}^\dagger(\theta, \phi) \quad (\text{D.3})$$

D.2.1 Orthogonality and completeness

The spherical harmonics are orthonormal over the unit sphere Ω , *i.e.*,

$$\iint_{\Omega} Y_{lm}(\hat{\mathbf{r}}) Y_{l'm'}^\dagger(\hat{\mathbf{r}}) d\Omega = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta Y_{lm}(\theta, \phi) Y_{l'm'}^\dagger(\theta, \phi) = \delta_{ll'} \delta_{mm'}$$

Moreover, the system $Y_{lm}(\hat{\mathbf{r}})$, $l = 0, 1, \dots$, $m = -l, \dots, 0, \dots, l$, is a complete orthonormal system over the unit sphere, see [3, page 24]. A square integrable function $f(\theta, \phi)$ defined on the unit sphere has a convergent expansion in the spherical harmonics $Y_{lm}(\hat{\mathbf{r}})$.

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} Y_{lm}(\theta, \phi), \quad \theta \in [0, \pi], \phi \in [0, 2\pi)$$

where the generalized Fourier-coefficients a_{lm} are determined by the integrals

$$a_{lm} = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta f(\theta, \phi) Y_{lm}^\dagger(\theta, \phi)$$

¹Therefore, this symbol may seem needless, but in combination with the radial functions forming the spherical vector waves in Chapter 2, the notion becomes handy.

The convergence is in the norm sense

$$\left\| f(\hat{\mathbf{r}}) - \sum_{l=0}^N \sum_{m=-l}^l a_{lm} Y_{lm}(\hat{\mathbf{r}}) \right\|^2 = \|f(\hat{\mathbf{r}})\|^2 - \sum_{l=0}^N \sum_{m=-l}^l |a_{lm}|^2 \rightarrow 0, \text{ da } N \rightarrow \infty$$

where the norm $\|\cdot\|$ is defined as

$$\|f(\hat{\mathbf{r}})\|^2 = \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta f(\hat{\mathbf{r}}) f^*(\hat{\mathbf{r}}) = \iint_{\Omega} |f(\hat{\mathbf{r}})|^2 d\Omega$$

The completeness can also be stated as

$$\int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta f(\theta, \phi) Y_{lm}^\dagger(\theta, \phi) = 0, \text{ for all } l, m \Rightarrow f(\theta, \phi) = 0$$

D.2.2 Vector operations on $Y_n(\hat{\mathbf{r}})$

The spherical harmonics $Y_{lm}(\hat{\mathbf{r}})$ are eigenfunctions to the Laplace operator ∇_{Ω}^2 on the unit sphere with eigenvalues $-l(l+1)$, *i.e.*,

$$\nabla_{\Omega}^2 Y_{lm}(\hat{\mathbf{r}}) = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} Y_{lm}(\hat{\mathbf{r}}) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} Y_{lm}(\hat{\mathbf{r}}) = -l(l+1) Y_{lm}(\hat{\mathbf{r}}) \quad (\text{D.4})$$

D.3 Vector spherical harmonics

A vector field $\mathbf{F}(\hat{\mathbf{r}})$ defined on the unit sphere has a series expansion in the spherical harmonics defined in Section D.2.

$$\mathbf{F}(\hat{\mathbf{r}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \mathbf{a}_{lm} Y_{lm}(\hat{\mathbf{r}})$$

The drawback with this expansion is that the expansion coefficients \mathbf{a}_{lm} are vector-valued. If there was a set of vector-valued basis function, the expansion coefficients would be scalar. To accomplish this, let us introduce the vector spherical harmonics.

The vector spherical harmonics can be defined in several ways. In this textbook we adopt the following definition:

$$\begin{cases} \mathbf{A}_{1lm}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}} \nabla \times (\mathbf{r} Y_{lm}(\hat{\mathbf{r}})) = \frac{1}{\sqrt{l(l+1)}} \nabla Y_{lm}(\hat{\mathbf{r}}) \times \mathbf{r} \\ \mathbf{A}_{2lm}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}} r \nabla Y_{lm}(\hat{\mathbf{r}}) \\ \mathbf{A}_{3lm}(\hat{\mathbf{r}}) = \hat{\mathbf{r}} Y_{lm}(\hat{\mathbf{r}}) \end{cases} \quad (\text{D.5})$$

The integer indices are $l = 0, 1, 2, 3, \dots$ and $m = -l, \dots, 0, 1, \dots, l$. For $l = 0$, the first two expressions are not defined, since both the nominator and the denominator

are zero. We define these vector spherical harmonics to be zero, *i.e.*, $\mathbf{A}_{100}(\hat{\mathbf{r}}) = \mathbf{A}_{200}(\hat{\mathbf{r}}) = \mathbf{0}$. Alternatively, we can express the vector spherical harmonics in the spherical basis vectors $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$. The result is

$$\begin{cases} \mathbf{A}_{1lm}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}} \left(\hat{\boldsymbol{\theta}} \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} Y_{lm}(\hat{\mathbf{r}}) - \hat{\boldsymbol{\phi}} \frac{\partial}{\partial \theta} Y_{lm}(\hat{\mathbf{r}}) \right) \\ \mathbf{A}_{2lm}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}} \left(\hat{\boldsymbol{\theta}} \frac{\partial}{\partial \theta} Y_{lm}(\hat{\mathbf{r}}) + \hat{\boldsymbol{\phi}} \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} Y_{lm}(\hat{\mathbf{r}}) \right) \\ \mathbf{A}_{3lm}(\hat{\mathbf{r}}) = \hat{\mathbf{r}} Y_{lm}(\hat{\mathbf{r}}) \end{cases}$$

In analogy with the spherical harmonics in Section D.2, we frequently need to change the sign in the exponential in the vector spherical harmonics, *i.e.*,

$$\begin{cases} \mathbf{A}_{1lm}^\dagger(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}} \nabla \times (\mathbf{r} Y_{lm}^\dagger(\hat{\mathbf{r}})) = \frac{1}{\sqrt{l(l+1)}} \nabla Y_{lm}^\dagger(\hat{\mathbf{r}}) \times \mathbf{r} \\ \mathbf{A}_{2lm}^\dagger(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}} r \nabla Y_{lm}^\dagger(\hat{\mathbf{r}}) \\ \mathbf{A}_{3lm}^\dagger(\hat{\mathbf{r}}) = \hat{\mathbf{r}} Y_{lm}^\dagger(\hat{\mathbf{r}}) \end{cases} \quad (\text{D.6})$$

We have

$$\mathbf{A}_n^\dagger(\theta, \phi) = \mathbf{A}_n^*(\theta, \phi), \quad \theta, \phi \in \mathbb{R}$$

and

$$\mathbf{A}_{\tau l - m}(\theta, \phi) = (-1)^m \mathbf{A}_{\tau l m}^\dagger(\theta, \phi)$$

These vector spherical harmonics have the following properties:

$$\begin{cases} \hat{\mathbf{r}} \cdot \mathbf{A}_{\tau l m}(\hat{\mathbf{r}}) = 0, \quad \tau = 1, 2 \\ \hat{\mathbf{r}} \times \mathbf{A}_{3 l m}(\hat{\mathbf{r}}) = \mathbf{0} \end{cases}$$

and

$$\begin{cases} \mathbf{A}_{1 l m}(\hat{\mathbf{r}}) = \mathbf{A}_{2 l m}(\hat{\mathbf{r}}) \times \hat{\mathbf{r}} \\ \mathbf{A}_{2 l m}(\hat{\mathbf{r}}) = \hat{\mathbf{r}} \times \mathbf{A}_{1 l m}(\hat{\mathbf{r}}) \end{cases}$$

which we can summarize as

$$\hat{\mathbf{r}} \times \mathbf{A}_n(\hat{\mathbf{r}}) = (-1)^{\tau+1} \mathbf{A}_{\bar{n}}(\hat{\mathbf{r}}), \quad \tau = 1, 2 \quad (\text{D.7})$$

where we have introduced the dual index to τ , defined by $\bar{1} = 2$ and $\bar{2} = 1$.

Inversion of the argument, $\hat{\mathbf{r}} \rightarrow -\hat{\mathbf{r}}$, *i.e.*, $\theta \rightarrow \pi - \theta$, $\phi \rightarrow \phi + \pi$, implies

$$\begin{cases} \mathbf{A}_{1 l m}(-\hat{\mathbf{r}}) = (-1)^l \mathbf{A}_{1 l m}(\hat{\mathbf{r}}) \\ \mathbf{A}_{2 l m}(-\hat{\mathbf{r}}) = (-1)^{l+1} \mathbf{A}_{2 l m}(\hat{\mathbf{r}}) \\ \mathbf{A}_{3 l m}(-\hat{\mathbf{r}}) = (-1)^{l+1} \mathbf{A}_{3 l m}(\hat{\mathbf{r}}) \end{cases} \quad (\text{D.8})$$

D.3.1 Specific values of argument and order

With an argument $\hat{\mathbf{r}} = \hat{\mathbf{z}}$ ($\theta = 0$), the vector spherical harmonics have specific values.

$$\begin{cases} \mathbf{A}_{1lm}(\hat{\mathbf{z}}) = \mp \delta_{m\pm 1} \sqrt{\frac{2l+1}{16\pi}} (\pm i\hat{\mathbf{x}} - \hat{\mathbf{y}}) = \pm \delta_{m\pm 1} \sqrt{\frac{2l+1}{16\pi}} \hat{\mathbf{z}} \times (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) \\ \mathbf{A}_{2lm}(\hat{\mathbf{z}}) = \mp \delta_{m\pm 1} \sqrt{\frac{2l+1}{16\pi}} (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) \\ \mathbf{A}_{3lm}(\hat{\mathbf{z}}) = \delta_{m0} \sqrt{\frac{2l+1}{4\pi}} \hat{\mathbf{z}} \end{cases} \quad (\text{D.9})$$

$m = 0$

Specific values for $m = 0$ are

$$\begin{cases} \mathbf{A}_{1l0}(\hat{\mathbf{r}}) = \sqrt{\frac{2l+1}{4\pi l(l+1)}} \hat{\phi} \sin \theta P'_l(\cos \theta) \\ \mathbf{A}_{2l0}(\hat{\mathbf{r}}) = -\sqrt{\frac{2l+1}{4\pi l(l+1)}} \hat{\theta} \sin \theta P'_l(\cos \theta) \\ \mathbf{A}_{3l0}(\hat{\mathbf{r}}) = \sqrt{\frac{2l+1}{4\pi}} \hat{\mathbf{r}} P_l(\cos \theta) \end{cases}$$

The lowest order vector spherical harmonics for $m = 0$ can be written explicitly as

$$\begin{cases} \mathbf{A}_{100}(\hat{\mathbf{r}}) = \mathbf{0} \\ \mathbf{A}_{200}(\hat{\mathbf{r}}) = \mathbf{0} \\ \mathbf{A}_{300}(\hat{\mathbf{r}}) = \sqrt{\frac{1}{4\pi}} \hat{\mathbf{r}} \end{cases}$$

for $l = 0$, and similarly for $l = 1$

$$\begin{cases} \mathbf{A}_{110}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{8\pi}} \hat{\phi} \sin \theta = -\sqrt{\frac{3}{8\pi}} \hat{\mathbf{r}} \times \hat{\mathbf{z}} \\ \mathbf{A}_{210}(\hat{\mathbf{r}}) = -\sqrt{\frac{3}{8\pi}} \hat{\theta} \sin \theta = \sqrt{\frac{3}{8\pi}} (\hat{\mathbf{z}} - \hat{\mathbf{r}} \cos \theta) = \sqrt{\frac{3}{8\pi}} \hat{\mathbf{r}} \times (\hat{\mathbf{z}} \times \hat{\mathbf{r}}) \\ \mathbf{A}_{310}(\hat{\mathbf{r}}) = \sqrt{\frac{3}{4\pi}} \hat{\mathbf{r}} \cos \theta = \sqrt{\frac{3}{4\pi}} \hat{\mathbf{r}} (\hat{\mathbf{z}} \cdot \hat{\mathbf{r}}) \end{cases} \quad (\text{D.10})$$

Since $\hat{\mathbf{z}} = \hat{\mathbf{r}} \cos \theta - \hat{\theta} \sin \theta$, this result implies that the unit vector $\hat{\mathbf{z}}$ can be expressed in the vector spherical harmonics for $l = 1$ and $m = 0$.

$$\hat{\mathbf{z}} = \sqrt{\frac{8\pi}{3}} \left(\mathbf{A}_{210}(\hat{\mathbf{r}}) + \frac{1}{\sqrt{2}} \mathbf{A}_{310}(\hat{\mathbf{r}}) \right)$$

$m = \pm 1$

Moreover, the vector spherical harmonics for $m = \pm 1$ are

$$\left\{ \begin{array}{l} \mathbf{A}_{1l\pm 1}(\hat{\mathbf{r}}) = \mp \frac{1}{l(l+1)} \sqrt{\frac{2l+1}{4\pi}} \left(\pm i \hat{\boldsymbol{\theta}} P'_l(\cos \theta) \right. \\ \quad \left. + \hat{\boldsymbol{\phi}} (\cos \theta P'_l(\cos \theta) - l(l+1)P_l(\cos \theta)) \right) e^{\pm i\phi} \\ \mathbf{A}_{2l\pm 1}(\hat{\mathbf{r}}) = \mp \frac{1}{l(l+1)} \sqrt{\frac{2l+1}{4\pi}} \left(\pm i \hat{\boldsymbol{\phi}} P'_l(\cos \theta) \right. \\ \quad \left. - \hat{\boldsymbol{\theta}} (\cos \theta P'_l(\cos \theta) - l(l+1)P_l(\cos \theta)) \right) e^{\pm i\phi} \\ \mathbf{A}_{3l\pm 1}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{2l+1}{4\pi l(l+1)}} \hat{\mathbf{r}} \sin \theta P'_l(\cos \theta) e^{\pm i\phi} \end{array} \right. \quad (\text{D.11})$$

Specifically, we get for $l = 1$

$$\left\{ \begin{array}{l} \mathbf{A}_{11\pm 1}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{3}{16\pi}} (\pm i \hat{\boldsymbol{\theta}} - \hat{\boldsymbol{\phi}} \cos \theta) e^{\pm i\phi} = \mp \sqrt{\frac{3}{16\pi}} (\hat{\mathbf{x}} \pm i \hat{\mathbf{y}}) \times \hat{\mathbf{r}} \\ \mathbf{A}_{21\pm 1}(\hat{\mathbf{r}}) = \hat{\mathbf{r}} \times \mathbf{A}_{11\pm 1}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{3}{16\pi}} \hat{\mathbf{r}} \times ((\hat{\mathbf{x}} \pm i \hat{\mathbf{y}}) \times \hat{\mathbf{r}}) \\ \mathbf{A}_{31\pm 1}(\hat{\mathbf{r}}) = \mp \sqrt{\frac{3}{8\pi}} \hat{\mathbf{r}} \sin \theta e^{\pm i\phi} = \mp \sqrt{\frac{3}{8\pi}} \hat{\mathbf{r}} ((\hat{\mathbf{x}} \pm i \hat{\mathbf{y}}) \cdot \hat{\mathbf{r}}) \end{array} \right. \quad (\text{D.12})$$

D.3.2 Vector operations on $\mathbf{A}_n(\hat{\mathbf{r}})$

The divergence and the curl of these relations, obtained from the definitions of the vector spherical harmonics (D.5) and the use of the rules of the ∇ -operator, are

$$\left\{ \begin{array}{l} \nabla \cdot \mathbf{A}_{1lm}(\hat{\mathbf{r}}) = 0 \\ \nabla \cdot \mathbf{A}_{2lm}(\hat{\mathbf{r}}) = -\sqrt{l(l+1)} \frac{Y_{lm}(\hat{\mathbf{r}})}{r} \\ \nabla \cdot \mathbf{A}_{3lm}(\hat{\mathbf{r}}) = \frac{2Y_{lm}(\hat{\mathbf{r}})}{r} \end{array} \right. \quad (\text{D.13})$$

and²

$$\left\{ \begin{array}{l} \nabla \times \mathbf{A}_{1lm}(\hat{\mathbf{r}}) = \frac{\mathbf{A}_{2lm}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \mathbf{A}_{3lm}(\hat{\mathbf{r}})}{r} \\ \nabla \times \mathbf{A}_{2lm}(\hat{\mathbf{r}}) = -\frac{\mathbf{A}_{1lm}(\hat{\mathbf{r}})}{r} \\ \nabla \times \mathbf{A}_{3lm}(\hat{\mathbf{r}}) = \frac{\sqrt{l(l+1)} \mathbf{A}_{1lm}(\hat{\mathbf{r}})}{r} \end{array} \right. \quad (\text{D.14})$$

²We have here used $\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b}$ with $\mathbf{a} = \nabla Y_{lm}(\hat{\mathbf{r}})$ and $\mathbf{b} = \mathbf{r}$, and that $\nabla \cdot \mathbf{r} = 3$, $\nabla^2 Y_{lm}(\hat{\mathbf{r}}) = -r^{-2} l(l+1) Y_{lm}(\hat{\mathbf{r}})$, $\frac{\partial}{\partial r} (\nabla Y_{lm}(\hat{\mathbf{r}})) = -r^{-1} \nabla Y_{lm}(\hat{\mathbf{r}})$ and $(\mathbf{a} \cdot \nabla) \mathbf{r} = \mathbf{a}$ for an arbitrary vector \mathbf{a} .

Using these relations, we obtain the double curl of the vector spherical harmonics

$$\begin{cases} \nabla \times (\nabla \times \mathbf{A}_{1lm}(\hat{\mathbf{r}})) = \frac{l(l+1)\mathbf{A}_{1lm}(\hat{\mathbf{r}})}{r^2} \\ \nabla \times (\nabla \times \mathbf{A}_{2lm}(\hat{\mathbf{r}})) = -\frac{\sqrt{l(l+1)}\mathbf{A}_{3lm}(\hat{\mathbf{r}})}{r^2} \\ \nabla \times (\nabla \times \mathbf{A}_{3lm}(\hat{\mathbf{r}})) = \frac{l(l+1)\mathbf{A}_{3lm}(\hat{\mathbf{r}})}{r^2} \end{cases}$$

and that

$$\begin{cases} \nabla_{\Omega}^2 \mathbf{A}_{1lm}(\hat{\mathbf{r}}) = -l(l+1)\mathbf{A}_{1lm}(\hat{\mathbf{r}}) \\ \nabla_{\Omega}^2 \mathbf{A}_{2lm}(\hat{\mathbf{r}}) = -l(l+1)\mathbf{A}_{2lm}(\hat{\mathbf{r}}) + 2\sqrt{l(l+1)}\mathbf{A}_{3lm}(\hat{\mathbf{r}}) \\ \nabla_{\Omega}^2 \mathbf{A}_{3lm}(\hat{\mathbf{r}}) = -(l(l+1)+2)\mathbf{A}_{3lm}(\hat{\mathbf{r}}) + 2\sqrt{l(l+1)}\mathbf{A}_{2lm}(\hat{\mathbf{r}}) \end{cases} \quad (\text{D.15})$$

where ∇_{Ω}^2 denotes the Laplace operator on the unit sphere, *i.e.*,

$$\nabla_{\Omega}^2 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

D.3.3 Orthogonality and completeness

The vector spherical harmonics satisfy the orthogonal relation on the unit sphere.

$$\begin{aligned} \iint_{\Omega} \mathbf{A}_{\tau lm}(\hat{\mathbf{r}}) \cdot \mathbf{A}_{\tau' l' m'}^{\dagger}(\hat{\mathbf{r}}) \, d\Omega \\ = \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta \, d\theta \, \mathbf{A}_{\tau lm}(\theta, \phi) \cdot \mathbf{A}_{\tau' l' m'}^{\dagger}(\theta, \phi) = \delta_{\tau\tau'} \delta_{ll'} \delta_{mm'} \end{aligned}$$

The natural norm for vector-valued functions on the unit sphere is $\|\cdot\|$.

$$\|\mathbf{F}(\hat{\mathbf{r}})\|^2 = \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta \, d\theta \, \mathbf{F}(\hat{\mathbf{r}}) \cdot \mathbf{F}^*(\hat{\mathbf{r}}) = \iint_{\Omega} |\mathbf{F}(\hat{\mathbf{r}})|^2 \, d\Omega$$

Every square integrable vector field defined on the unit sphere has a norm-convergent expansion in vector spherical harmonics. The completeness of these basis functions is found in, *e.g.*, [3, page 170]. The vector field $\mathbf{F}(\theta, \phi)$ defined on the unit sphere is expanded as

$$\mathbf{F}(\theta, \phi) = \sum_{\tau=1}^3 \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{\tau lm} \mathbf{A}_{\tau lm}(\theta, \phi), \quad \theta \in [0, \pi], \phi \in [0, 2\pi)$$

where the Fourier-coefficients $a_{\tau lm}$ are determined by the integrals

$$a_{\tau lm} = \iint_{\Omega} \mathbf{F}(\hat{\mathbf{r}}) \cdot \mathbf{A}_{\tau lm}^{\dagger}(\hat{\mathbf{r}}) \, d\Omega$$

Note that the expansion coefficients in this expansion are scalars. The convergence is in general only in the norm sense.

$$\left\| \mathbf{F}(\hat{\mathbf{r}}) - \sum_{\tau=1}^3 \sum_{l=0}^N \sum_{m=-l}^l a_{\tau lm} \mathbf{A}_{\tau lm}(\hat{\mathbf{r}}) \right\|^2 = \|\mathbf{F}(\hat{\mathbf{r}})\|^2 - \sum_{\tau=1}^3 \sum_{l=0}^N \sum_{m=-l}^l |a_{\tau lm}|^2 \rightarrow 0, \text{ d\aa } N \rightarrow \infty$$

D.4 Addition theorem for the Legendre polynomials

In this section we prove the addition theorem for the Legendre polynomials $P_l(\cos \gamma)$.

$$P_l(\cos \gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^\dagger(\hat{\mathbf{k}}) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}^\dagger(\hat{\mathbf{r}}) Y_{lm}(\hat{\mathbf{k}})$$

where

$$\cos \gamma = \hat{\mathbf{r}} \cdot \hat{\mathbf{k}} = \sin \theta \sin \alpha \cos(\phi - \beta) + \cos \theta \cos \alpha$$

and (θ, ϕ) and (α, β) are the spherical angles for the unit vectors $\hat{\mathbf{r}}$ and $\hat{\mathbf{k}}$, respectively, see Figure D.2.

To prove this formula we start by expanding an arbitrary function $f(\hat{\mathbf{r}})$ in terms of spherical harmonics.

$$f(\hat{\mathbf{r}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} Y_{lm}(\hat{\mathbf{r}})$$

where

$$a_{lm} = \iint_{\Omega} f(\hat{\mathbf{r}}) Y_{lm}^\dagger(\hat{\mathbf{r}}) \, d\Omega$$

and where $d\Omega = \sin \theta \, d\theta \, d\phi$. Specifically, we have for any solution $f_l(\hat{\mathbf{r}})$ of the equation (l non-negative integer)

$$r^2 \nabla^2 f_l(\hat{\mathbf{r}}) = -l(l+1) f_l(\hat{\mathbf{r}}) \tag{D.16}$$

an expansion

$$f_l(\hat{\mathbf{r}}) = \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} a_{l'm'} Y_{l'm'}(\hat{\mathbf{r}})$$

In this expansion only $l' = l$ contributes. To see this, form the quantity $r^2 \nabla^2 f_l(\hat{\mathbf{r}})$ in two ways. First by operating $r^2 \nabla^2$ on f_l and use the differential equation that

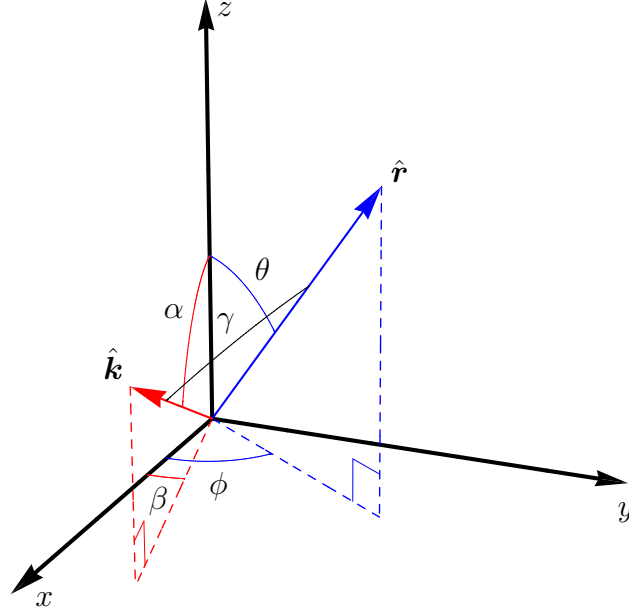


Figure D.2: The definition of the spherical angles (θ, ϕ) , (α, β) and the angle γ . The unit vector $\hat{\mathbf{r}}$ and its spherical angles are given in blue lines, and the unit vector $\hat{\mathbf{k}}$ and its spherical angles are given in red lines.

f_l satisfies, and then secondly, by differentiating under the summation signs. By equating these two expressions we get

$$-l(l+1) \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} a_{l'm'} Y_{l'm'}(\hat{\mathbf{r}}) = - \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} l'(l'+1) a_{l'm'} Y_{l'm'}(\hat{\mathbf{r}})$$

where we used (D.4). Since the expansion coefficients $a_{l'm'}$ are uniquely defined, we conclude that they are zero unless³ $l' = l$, *i.e.*,

$$f_l(\hat{\mathbf{r}}) = \sum_{m=-l}^l a_{lm} Y_{lm}(\hat{\mathbf{r}})$$

All functions $f_l(\hat{\mathbf{r}})$ satisfying (D.16) are in other words expressed as an expansion of spherical harmonics of order l . Especially, evaluate this expansion at $\theta = 0$ ($\hat{\mathbf{r}} = \hat{\mathbf{z}}$). We get, since $Y_{lm}(\theta = 0, \phi) = \sqrt{(2l+1)/4\pi} \delta_{m0}$

$$f_l(\hat{\mathbf{r}})|_{\theta=0} = \sum_{m=-l}^l a_{lm} Y_{lm}(\theta = 0, \phi) = \sqrt{\frac{2l+1}{4\pi}} a_{l0}$$

³The other possibility $l' = -l - 1$ is ruled out since l' non-negative integer.

where

$$a_{10} = \iint_{\Omega} f_l(\hat{\mathbf{r}}) Y_{10}^\dagger(\hat{\mathbf{r}}) \, d\Omega = \sqrt{\frac{2l+1}{4\pi}} \iint_{\Omega} f_l(\hat{\mathbf{r}}) P_l(\cos \theta) \, d\Omega$$

Therefore,

$$\iint_{\Omega} f_l(\hat{\mathbf{r}}) P_l(\cos \theta) \, d\Omega = \frac{4\pi}{2l+1} f_l(\hat{\mathbf{r}})|_{\theta=0} \quad (\text{D.17})$$

We now apply this result to $P_l(\cos \gamma)$, which satisfies (D.16). We get an expansion

$$P_l(\cos \gamma) = \sum_{m=-l}^l a_{lm} Y_{lm}(\theta, \phi)$$

where

$$a_{lm} = \iint_{\Omega} P_l(\cos \gamma) Y_{lm}^\dagger(\theta, \phi) \, d\Omega(\theta, \phi) = \iint_{\Omega} P_l(\cos \gamma) Y_{lm}^\dagger(\theta, \phi) \, d\Omega(\gamma, \psi)$$

We have here used the fact that the surface measure, $d\Omega(\theta, \phi) = \sin \theta \, d\theta \, d\phi$, on the unit sphere is invariant under rotations, and therefore the same as for the rotated coordinate system $d\Omega(\gamma, \psi) = \sin \gamma \, d\gamma \, d\psi$. From the result above, see (D.17), this integral can be evaluated as (use $f_l(\hat{\mathbf{r}}) = Y_{lm}^\dagger(\theta, \phi)$)

$$a_{lm} = \frac{4\pi}{2l+1} Y_{lm}^\dagger(\theta, \phi) \Big|_{\gamma=0} = \frac{2l+1}{4\pi} Y_{lm}^\dagger(\hat{\mathbf{k}})$$

and the final result is

$$\boxed{P_l(\cos \gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^\dagger(\hat{\mathbf{k}}) = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}^\dagger(\hat{\mathbf{r}}) Y_{lm}(\hat{\mathbf{k}})} \quad (\text{D.18})$$

The last equality is obtained by changing the summation index m to $-m$ and using (D.3).

In particular, we get for $\hat{\mathbf{r}} = \hat{\mathbf{k}}$

$$\sum_{m=-l}^l Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^\dagger(\hat{\mathbf{r}}) = \frac{2l+1}{4\pi}$$

which is independent of the direction $\hat{\mathbf{r}}$.

D.5 Transformation formulae

We can use (B.20), *i.e.*,

$$j_l(x) = \frac{1}{2i^l} \int_{-1}^1 P_l(t) e^{itx} \, dt$$

to establish the transformation between a plane wave with propagation direction $\hat{\mathbf{k}}$, *i.e.*, $e^{i\hat{\mathbf{k}}\cdot\mathbf{r}}$, and a regular spherical wave, $j_l(kr)Y_{lm}(\hat{\mathbf{r}})$ and its inverse. We start by expanding $\exp\{ikz\}$ in a Legendre series

$$e^{ikz} = e^{ikr \cos \theta} = \sum_{l=0}^{\infty} a_l P_l(\cos \theta)$$

The coefficients a_l in this expansion is found by

$$a_l = \frac{2l+1}{2} \int_0^\pi e^{ikr \cos \theta} P_l(\cos \theta) \sin \theta \, d\theta = i^l (2l+1) j_l(kr)$$

where we have used the identity (B.20). Thus, we have

$$e^{ikr \cos \theta} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos \theta)$$

Introduce the wave vector $\mathbf{k} = k\hat{\mathbf{k}}$ and the position vector $\mathbf{r} = r\hat{\mathbf{r}}$ and let $\cos \gamma = \hat{\mathbf{k}}\cdot\hat{\mathbf{r}}$. Then

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos \gamma) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l i^l j_l(kr) Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^\dagger(\hat{\mathbf{k}})$$

where the addition theorem for the Legendre polynomial, (D.18), is used. This is the transformation between a regular spherical wave and a plane wave.

The inverse of this transformation is also obtained by multiplication by $Y_{lm}(\hat{\mathbf{k}})$, and integration over the unit sphere in the $\hat{\mathbf{k}}$ -variables. Orthogonality gives

$$j_l(kr) Y_{lm}(\hat{\mathbf{r}}) = \frac{1}{4\pi i^l} \iint_{\Omega} e^{i\mathbf{k}\cdot\mathbf{r}} Y_{lm}(\hat{\mathbf{k}}) \, d\Omega_{\hat{\mathbf{k}}} \quad (\text{D.19})$$

We summarize,

$$\boxed{\begin{aligned} e^{i\mathbf{k}\cdot\mathbf{r}} &= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l i^l j_l(kr) Y_{lm}(\hat{\mathbf{r}}) Y_{lm}^\dagger(\hat{\mathbf{k}}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l i^l j_l(kr) Y_{lm}^\dagger(\hat{\mathbf{r}}) Y_{lm}(\hat{\mathbf{k}}) \\ j_l(kr) Y_{lm}(\hat{\mathbf{r}}) &= \frac{1}{4\pi i^l} \iint_{\Omega} e^{i\mathbf{k}\cdot\mathbf{r}} Y_{lm}(\hat{\mathbf{k}}) \, d\Omega_{\hat{\mathbf{k}}} \end{aligned}}$$

The first relation transforms a plane wave to a regular spherical wave. The second relation is the transformation of a regular spherical wave to a plane wave.

∇ in curvilinear coordinate systems

In this appendix some important expressions with the ∇ -operator in two curvilinear coordinate systems, cylindrical and spherical, are collected. For completeness we start with the Cartesian coordinate system.

E.1 Cartesian coordinate system

The Cartesian coordinates (x, y, z) is the most basic coordinate system. The gradient and the Laplace-operator of a scalar field $\psi(x, y, z)$ in this coordinate system are

$$\begin{aligned}\nabla\psi &= \hat{\mathbf{x}}\frac{\partial\psi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\psi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\psi}{\partial z} \\ \nabla^2\psi &= \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\end{aligned}$$

The divergence, the curl, and the Laplace-operator of a vector field $\mathbf{A}(x, y, z) = \hat{\mathbf{x}}A_x(x, y, z) + \hat{\mathbf{y}}A_y(x, y, z) + \hat{\mathbf{z}}A_z(x, y, z)$ are

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ \nabla \times \mathbf{A} &= \hat{\mathbf{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\ \nabla^2 \mathbf{A} &= \hat{\mathbf{x}} \nabla^2 A_x + \hat{\mathbf{y}} \nabla^2 A_y + \hat{\mathbf{z}} \nabla^2 A_z\end{aligned}$$

E.2 Circular cylindrical (polar) coordinate system

We now treat the first curvilinear coordinate system, and start with the circular cylindrical coordinate system (ρ, ϕ, z) defined by

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \begin{cases} \arccos \frac{x}{\sqrt{x^2 + y^2}} & y \geq 0 \\ 2\pi - \arccos \frac{x}{\sqrt{x^2 + y^2}} & y < 0 \end{cases} \\ z = z \end{cases}$$

The gradient and the Laplace-operator of a scalar field $\psi(\rho, \phi, z)$ in this coordinate system are

$$\begin{aligned} \nabla\psi &= \hat{\rho} \frac{\partial\psi}{\partial\rho} + \hat{\phi} \frac{1}{\rho} \frac{\partial\psi}{\partial\phi} + \hat{z} \frac{\partial\psi}{\partial z} \\ \nabla^2\psi &= \frac{1}{\rho} \frac{\partial}{\partial\rho} \left(\rho \frac{\partial\psi}{\partial\rho} \right) + \frac{1}{\rho^2} \frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \end{aligned}$$

The divergence, the curl, and the Laplace-operator of a vector field $\mathbf{A}(\rho, \phi, z) = \hat{\rho}A_\rho(\rho, \phi, z) + \hat{\phi}A_\phi(\rho, \phi, z) + \hat{z}A_z(\rho, \phi, z)$ are

$$\begin{aligned} \nabla \cdot \mathbf{A} &= \frac{1}{\rho} \frac{\partial}{\partial\rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \\ \nabla \times \mathbf{A} &= \hat{\rho} \left(\frac{1}{\rho} \frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z} \right) + \hat{\phi} \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho} \right) + \hat{z} \frac{1}{\rho} \left(\frac{\partial}{\partial\rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi} \right) \\ \nabla^2 \mathbf{A} &= \hat{\rho} \left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial A_\phi}{\partial\phi} \right) + \hat{\phi} \left(\nabla^2 A_\phi - \frac{A_\phi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial A_\rho}{\partial\phi} \right) + \hat{z} \nabla^2 A_z \end{aligned}$$

E.3 Spherical coordinates system

The spherical coordinate system (r, θ, ϕ) (polar angle θ and the azimuth angle ϕ) is defined by

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \phi = \begin{cases} \arccos \frac{x}{\sqrt{x^2 + y^2}} & y \geq 0 \\ 2\pi - \arccos \frac{x}{\sqrt{x^2 + y^2}} & y < 0 \end{cases} \end{cases}$$

The gradient and the Laplace-operator of a scalar field $\psi(r, \theta, \phi)$ in this coordinate system are

$$\begin{aligned}\nabla\psi &= \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \\ \nabla^2\psi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \\ &= \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\end{aligned}$$

and the divergence, the curl, and the Laplace-operator of a vector field $\mathbf{A}(r, \theta, \phi) = \hat{\mathbf{r}}A_r(r, \theta, \phi) + \hat{\boldsymbol{\theta}}A_\theta(r, \theta, \phi) + \hat{\boldsymbol{\phi}}A_\phi(r, \theta, \phi)$ are

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \\ \nabla \times \mathbf{A} &= \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right) \\ &\quad + \hat{\boldsymbol{\theta}}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right) + \hat{\boldsymbol{\phi}}\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right) \\ \nabla^2 \mathbf{A} &= \hat{\mathbf{r}}\left(\nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2\cot\theta}{r^2}A_\theta - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right) \\ &\quad + \hat{\boldsymbol{\theta}}\left(\nabla^2 A_\theta - \frac{A_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}\right) \\ &\quad + \hat{\boldsymbol{\phi}}\left(\nabla^2 A_\phi - \frac{A_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}\right)\end{aligned}$$

Notation

Appropriate notation leads to a more easily understood, systematic, and structured text, and, in the same token, implies a tendency of making less errors and slips. Most of the notation is explained at the place in the text where they are introduced, but some more general notation that is often used is collected in this appendix.

- Vector-valued quantities (mostly in \mathbb{R}^3) is denoted in slanted bold face, *e.g.*, \mathbf{a} and \mathbf{b} , and vectors of unit length have a “hat” or caret (^) over a symbol, *e.g.*, $\hat{\mathbf{x}}$ and $\hat{\rho}$.
- The (Euclidean) scalar product between two vectors, \mathbf{a} and \mathbf{b} , is denoted in the usual standard way by a dot (\cdot), *i.e.*, $\mathbf{a} \cdot \mathbf{b}$. If the vectors are complex-valued the appropriate scalar product is $\mathbf{a}^* \cdot \mathbf{b}$, where the star $*$ denotes the complex conjugate of the vector. For complex-valued functions ($L_2(\Omega)$ -functions defined on a domain Ω) the scalar product is defined in the standard way

$$(f, g) = \iiint_{\Omega} f(\mathbf{r})g^*(\mathbf{r}) \, dv$$

- We make a distinction between a vector \mathbf{a} and its representation in components in a specific coordinate system, and denote the components as a column vector or with brackets around the vector, *i.e.*,

$$[\mathbf{a}] = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

where

$$\mathbf{a} = \hat{\mathbf{x}}a_x + \hat{\mathbf{y}}a_y + \hat{\mathbf{z}}a_z$$

- Linear vector-valued transformations (dyadics) are denoted in bold roman fonts, *e.g.*, \mathbf{A} . A linear transformation \mathbf{A} acting on a vector field \mathbf{a} gives a new vector field \mathbf{b} and we use the notation

$$\mathbf{b} = \mathbf{A} \cdot \mathbf{a}$$

In a specific coordinate system the linear transformation \mathbf{A} is represented by a 3×3 matrix $[\mathbf{A}]$, where we again use brackets around \mathbf{A} to emphasize that we refer to its components. The components of the vector \mathbf{b} is then

$$[\mathbf{b}] = [\mathbf{A}] \cdot [\mathbf{a}]$$

or

$$\begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

- The unity and the null dyadics in n dimensions are denoted \mathbf{I}_n and \mathbf{O}_n respectively, and the corresponding matrix representations are denoted $[\mathbf{I}]_n$ and $[\mathbf{O}]_n$, respectively. In three dimensions we have

$$[\mathbf{I}]_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad [\mathbf{O}]_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

or in two dimensions

$$[\mathbf{I}]_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad [\mathbf{O}]_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- The transpose of a matrix is denoted by a superscript $(^t)$ and the Hermitian of a matrix with the superscript dagger $(^\dagger)$, *i.e.*,

$$\begin{aligned} A_{ij}^t &= A_{ji} \\ A_{ij}^\dagger &= A_{ji}^* \end{aligned}$$

- We use the symbols o and O defined by

$$\begin{cases} f(x) = o(g(x)), & x \rightarrow a \Leftrightarrow \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 0 \\ f(x) = O(g(x)), & x \rightarrow a \Leftrightarrow \frac{f(x)}{g(x)} \text{ bounded in a neighborhood of } a \end{cases}$$

- The symbol \blacksquare denotes the end of an example.
- The real and the imaginary part of a complex number $z = x + iy$ are denoted $\text{Re } z$ and $\text{Im } z$, respectively, dvs.

$$\begin{aligned} \text{Re } z &= x = \frac{1}{2}(z + z^*) \\ \text{Im } z &= y = \frac{1}{2i}(z - z^*) \end{aligned}$$

A star $(^*)$ is used to denote the complex conjugate of a complex number, *i.e.*, $z^* = x - iy$.

- The Heaviside's step function, $H(t)$, is defined in the usual way as

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

- The Kronecker's delta (function) symbol, δ_{ij} , is defined as

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

- The cylindrical coordinate system (ρ, ϕ, z) is defined by

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}} & y \geq 0 \\ 2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}} & y < 0 \end{cases} \\ z = z \end{cases}$$

The domain of the coordinates are $\rho \in [0, \infty)$, $\phi \in [0, 2\pi)$, and $z \in (-\infty, \infty)$.

- The spherical coordinate system (r, θ, ϕ) is defined as

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \phi = \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}} & y \geq 0 \\ 2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}} & y < 0 \end{cases} \end{cases}$$

The domain of the coordinates are $r \in [0, \infty)$, $\theta \in [0, \pi]$, and $\phi \in [0, 2\pi)$.

Appendix G

Units and constants

The explicit form of the equations in electromagnetism varies depending on the system of units that we use. The SI-system is the one that is used in most literature nowadays, and this textbook is no exception. The relevant constant in the SI-system that is used in the text is collected in this appendix.

The speed of light in vacuum c_0 has the value (exact value)

$$c_0 = 299\,792\,458 \text{ m/s}$$

μ_0 and ϵ_0 denote the permeability and the permittivity of vacuum, respectively. Their exact values are

$$\begin{aligned}\mu_0 &= 4\pi \cdot 10^{-7} \text{ N/A}^2 \\ \epsilon_0 &= \frac{1}{c_0^2 \mu_0} \text{ F/m}\end{aligned}$$

Approximative values of these constants are

$$\begin{aligned}\mu_0 &\approx 12.566\,370\,614 \cdot 10^{-7} \text{ N/A}^2 \\ \epsilon_0 &\approx 8.854\,187\,817 \cdot 10^{-12} \text{ F/m}\end{aligned}$$

The wave impedance of vacuum is denoted

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = c_0 \mu_0 = 299\,792\,458 \cdot 4\pi \cdot 10^{-7} \Omega \approx 376.730\,314 \Omega$$

The charge of the electron, $-e$, and its mass, m , have the values

$$\begin{aligned}e &\approx 1.602\,177\,33 \cdot 10^{-19} \text{ C} \\ m &\approx 9.109\,389\,8 \cdot 10^{-31} \text{ kg} \\ e/m &\approx 1.758\,819\,63 \cdot 10^{11} \text{ C/kg}\end{aligned}$$

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Answers to problems

1.1 $\mathbf{E}(r, \omega)$ and $\mathbf{H}(r, \omega)$ are zero for $r > a$.

1.2

$$\mathbf{E} = \frac{pk^2}{\epsilon_0} \frac{e^{ikr}}{4\pi r} \left\{ [3\hat{\mathbf{r}}(\hat{\mathbf{z}} \cdot \hat{\mathbf{r}}) - \hat{\mathbf{z}}] \left(\frac{1}{k^2 r^2} - \frac{i}{kr} \right) + \hat{\mathbf{r}} \times (\hat{\mathbf{z}} \times \hat{\mathbf{r}}) \right\}$$

$$\mathbf{H} = -ik\omega p \frac{e^{ikr}}{4\pi r} \left(i - \frac{1}{kr} \right) \hat{\mathbf{r}} \times \hat{\mathbf{z}}$$

$$\langle \mathbf{S}(t) \rangle = \frac{|p|^2 k^3 \omega}{32\pi^2 r^2 \epsilon_0} \hat{\mathbf{r}} (1 - (\hat{\mathbf{z}} \cdot \hat{\mathbf{r}})^2)$$

$$P = \frac{|p|^2 k^3 \omega}{12\pi \epsilon_0}$$

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Transformation of unit vectors

Cylindrical coordinates (ρ, ϕ, z)

Spherical coordinates (r, θ, ϕ)

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \begin{cases} \arccos \frac{x}{\sqrt{x^2 + y^2}} & y > 0 \\ 2\pi - \arccos \frac{x}{\sqrt{x^2 + y^2}} & y < 0 \end{cases} \\ z = z \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \phi = \begin{cases} \arccos \frac{x}{\sqrt{x^2 + y^2}} & y > 0 \\ 2\pi - \arccos \frac{x}{\sqrt{x^2 + y^2}} & y < 0 \end{cases} \end{cases}$$

$$(r, \theta, \phi) \longrightarrow (x, y, z)$$

$$\begin{cases} \hat{r} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta \\ \hat{\theta} = \hat{x} \cos \theta \cos \phi + \hat{y} \cos \theta \sin \phi - \hat{z} \sin \theta \\ \hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi \end{cases}$$

$$(x, y, z) \longrightarrow (r, \theta, \phi)$$

$$\begin{cases} \hat{x} = \hat{r} \sin \theta \cos \phi + \hat{\theta} \cos \theta \cos \phi - \hat{\phi} \sin \phi \\ \hat{y} = \hat{r} \sin \theta \sin \phi + \hat{\theta} \cos \theta \sin \phi + \hat{\phi} \cos \phi \\ \hat{z} = \hat{r} \cos \theta - \hat{\theta} \sin \theta \end{cases}$$

$$(\rho, \phi, z) \longrightarrow (x, y, z)$$

$$\begin{cases} \hat{\rho} = \hat{x} \cos \phi + \hat{y} \sin \phi = (\hat{x}x + \hat{y}y)/\sqrt{x^2 + y^2} \\ \hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi = (-\hat{x}y + \hat{y}x)/\sqrt{x^2 + y^2} \\ \hat{z} = \hat{z} \end{cases}$$

$$(x, y, z) \longrightarrow (\rho, \phi, z)$$

$$\begin{cases} \hat{x} = \hat{\rho} \cos \phi - \hat{\phi} \sin \phi \\ \hat{y} = \hat{\rho} \sin \phi + \hat{\phi} \cos \phi \\ \hat{z} = \hat{z} \end{cases}$$

$$(r, \theta, \phi) \longrightarrow (\rho, \phi, z)$$

$$\begin{cases} \hat{r} = \hat{\rho} \sin \theta + \hat{z} \cos \theta \\ \hat{\theta} = \hat{\rho} \cos \theta - \hat{z} \sin \theta \\ \hat{\phi} = \hat{\phi} \end{cases}$$

$$(\rho, \phi, z) \longrightarrow (r, \theta, \phi)$$

$$\begin{cases} \hat{\rho} = \hat{r} \sin \theta + \hat{\theta} \cos \theta \\ \hat{\phi} = \hat{\phi} \\ \hat{z} = \hat{r} \cos \theta - \hat{\theta} \sin \theta \end{cases}$$

Important vector identities

- (1) $(\mathbf{a} \times \mathbf{c}) \times (\mathbf{b} \times \mathbf{c}) = \mathbf{c}((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c})$
- (2) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$
- (3) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$
- (4) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$

Integration formulas

Stokes' theorem and analogous theorems

- (1)
$$\iint_S (\nabla \times \mathbf{A}) \cdot \hat{\nu} \, dS = \int_C \mathbf{A} \cdot d\mathbf{r}$$
- (2)
$$\iint_S \hat{\nu} \times \nabla \varphi \, dS = \int_C \varphi \, d\mathbf{r}$$
- (3)
$$\iint_S (\hat{\nu} \times \nabla) \times \mathbf{A} \, dS = \int_C d\mathbf{r} \times \mathbf{A}$$

Gauss' theorem and analogous theorems

- (1)
$$\iiint_V \nabla \cdot \mathbf{A} \, dv = \iint_S \mathbf{A} \cdot \hat{\nu} \, dS$$
- (2)
$$\iiint_V \nabla \varphi \, dv = \iint_S \varphi \hat{\nu} \, dS$$
- (3)
$$\iiint_V \nabla \times \mathbf{A} \, dv = \iint_S \hat{\nu} \times \mathbf{A} \, dS$$

Green's theorems

- (1)
$$\iiint_V (\psi \nabla^2 \varphi - \varphi \nabla^2 \psi) \, dv = \iint_S (\psi \nabla \varphi - \varphi \nabla \psi) \cdot \hat{\nu} \, dS$$
- (2)
$$\begin{aligned} \iiint_V (\psi \nabla^2 \mathbf{A} - \mathbf{A} \nabla^2 \psi) \, dv \\ = \iint_S (\nabla \psi \times (\hat{\nu} \times \mathbf{A}) - \nabla \psi (\hat{\nu} \cdot \mathbf{A}) - \psi (\hat{\nu} \times (\nabla \times \mathbf{A})) + \hat{\nu} \psi (\nabla \cdot \mathbf{A})) \, dS \end{aligned}$$

